GROWTH CURVES FOR ALGEBRAS

by

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Abstract. This paper studies matrix representations of algebras (over a field) using countably-infinite matrices which are both row and column finite, and in which the bandwidth growth is controlled. The ideas lead naturally to a concept of "growth of an algebra", somewhat analogous to the growth associated with GK–dimension. They also lead in a similar way to a dimension function on general algebras, which we term bandwidth dimension. For each real number $r \in [0,1]$, we construct an algebra having bandwidth dimension precisely r. Since the free algebra turns out to have bandwidth dimension 0, our new dimension promises to distinguish among algebras of infinite GK–dimension.

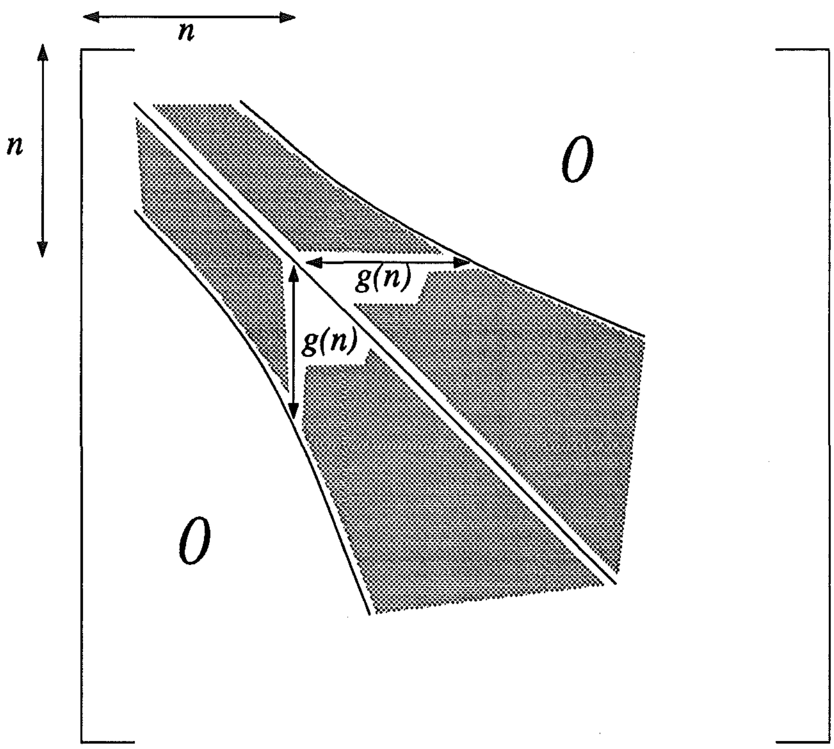
Introduction

One of the most basic facts in ring theory is that every algebra A over a field F can be (faithfully) represented as a subalgebra of the algebra $\mathrm{End}_F(V)$ of all linear transformations of some vector space V. For finite-dimensional algebras, everyone is happy to jump back and forth between representations in $\mathrm{End}_F(V)$ and matrix representations in $M_n(F)$ for $n = \dim V$, but it is probably fair to say that the matrix viewpoint is the dominant one. In the infinite-dimensional case, we still have a matrix representation of $\mathrm{End}_F(V)$ via column-finite matrices (assuming transformations are composed as $(f \circ g)(v) = f(g(v))$, otherwise we would need row-finite matrices). But even in the case of $\omega \times \omega$ column-finite matrices, some ring theorists feel a little bit uncomfortable with this type of representation. (Some analysts, of course, would treat it with scorn!) Certainly the matrix viewpoint is no longer the dominant one. However there are some important concepts in the finite-dimensional case which are naturally suggested by matrices but which are not so easily viewed in terms of transformations (for example, trace and character). It seems to us that the same is true in the infinite-dimensional case. The main idea in this paper, growth curves for algebras, comes literally from the visual impact of the matrix. (Nevertheless, the transformation viewpoint still has a valuable role in many of our proofs.)

The inspiration for our work has come directly from the surprising result in 1992 of Goodearl, Menal and Moncasi [GMM] that every countable-dimensional algebra A over F can be embedded in the algebra $B(F)$ of all $\omega \times \omega$ matrices which are simultaneously row-finite and column-finite. (The result was needed in [GMM] as an important step for establishing that, when F is countable, the free regular algebra over F on a countable set can be embedded in $\prod_{n \geq 1} M_n(F)$.) In such an embedding of A in $B(F)$, the elements of A have all their nonzero entries relatively close to the main diagonal, which raises the question of just how closely these nonzero entries can be squeezed to the main diagonal. This suggests the concept of a ***growth curve*** for an element of $B(F)$. We say that a function $g : \mathbf{N} \to \mathbf{R}^+$ is a growth curve for $x \in B(F)$ if for each $n \in \mathbf{N}$

$$x(n, i) = 0 = x(i, n)$$

for all $i > n + g(n)$. The picture below of the matrix x shows the motivation for this definition.



Every element x of $B(F)$ has such a growth curve (simply choose $g(n)$ so that all the entries of x in the n^{th} row and n^{th} column are zero more than $g(n)$ places beyond the diagonal). We say that $x \in B(F)$ has ***at most order*** $g(n)$ ***growth*** (or that x has $O(g(n))$ ***growth***), where $g : \mathbf{N} \to \mathbf{R}^+$, if there is some constant $c > 0$ such that the function $cg(n)$ is a growth curve for x. If A is a subalgebra of $B(F)$ and every $x \in A$ has $O(g(n))$ growth, then we say that the algebra A itself has $O(g(n))$ ***growth***. If A has $O(n)$ growth, then we say A has ***linear growth***.

Clearly $x \in B(F)$ can be chosen so that all its growth curves $g(n)$ increase as fast as we like. However we shall show that any countable-dimensional algebra A can be embedded in $B(F)$ as a subalgebra of linear growth (Theorem 2.1). In general this is the furthest that we can squeeze such representations of A. Indeed if A is purely infinite (that is $A \cong A \oplus A$ as right A–modules), then any representation of A in $B(F)$ contains an element whose growth curves must all satisfy $g(n) \geq n$ for infinitely many n (Theorem 3.3).

In that case, when is "sublinear" growth possible? We begin by identifying a range of sublinear growths. For $0 \leq r \leq 1$ we let $G(r)$ be the set of all $x \in B(F)$ having $O(n^r)$ growth. Then, as we shall see, each $G(r)$ is a subalgebra of $B(F)$. (If $r > 1$ then this construction does not give a subalgebra.) In terms of these subalgebras, the above results say that any countable-dimensional algebra A can be embedded in $G(1)$, while purely infinite algebras cannot be embedded in $G(r)$ for $r < 1$. This suggests the idea of using these indices r as a "dimension function" for algebras over F. If A is any countable-dimensional algebra over F, we define the

bandwidth dimension of A to be

$$\inf\{r \in \mathbf{R}, r \geq 0 \mid \text{there is an embedding of } A \text{ in } B(F) \text{ such that the image has } O(n^r) \text{ growth } \}$$

or, equivalently,

$$\inf\{r \in \mathbf{R}, r \geq 0 \mid A \text{ can be embedded in } G(r)\}.$$

By the linear growth result, the bandwidth dimension of countable-dimensional algebras takes values in $[0,1]$. We conjecture that all reals in $[0,1]$ can occur in this way (see §8).

For an uncountable-dimensional algebra A, the appropriate definition for its bandwidth dimension is not yet clear. The approach we shall adopt for the time being is to use the same definition as above but with the understanding that $\inf \Phi = \infty$. An alternative, but inequivalent, approach would be to mimic GK–dimension and define the bandwidth dimension of A as the supremum of the bandwidth dimensions of its countable-dimensional subalgebras. This alternative approach is possibly the more attractive and, hopefully, it won't be inconsistent with our approach for any algebra embeddable in $G(1)$.

With our adopted approach, the bandwidth dimension function for general algebras takes on all values in $[0,1]$. In fact, for any real number $r \in [0,1]$ we construct an algebra A of bandwidth dimension r, such that A is a subalgebra of $G(r)$ generated by a copy of a suitable $\prod_{k=1}^{\infty} M_{n_k}(F)$ and two additional elements (Theorem 8.8). A corollary is that $G(r)$ itself has bandwidth dimension r for each $r \in [0,1]$.

We conclude our introduction with some brief comments on the similarities and differences between bandwidth dimension and GK–dimension. (For a more detailed discussion, see the end of section 4.) That there are differences can be seen by considering the free algebra $F\{x,y\}$ on two generators. From the GK point of view, this algebra has exponential growth and therefore the *largest* possible GK–dimension, namely $+\infty$. In contrast, from our point of view, $F\{x,y\}$ embeds in the algebra $G(0)$ of finite bandwidth matrices and so has constant growth (Theorem 4.2). Accordingly, its bandwidth dimension is the *smallest* possible value, namely 0.

Nevertheless there are some similarities. Very roughly, one can view the GK–dimension of a finitely generated algebra A as determining the best *lower* growth curve for the generators of A, but only relative to the *regular* representation of A and then only relative to certain bases. The bandwidth dimension of A, on the other hand, determines the best *lower* and *upper* growth curve for these generators, relative to *all* matrix representations of A. (Indeed, even for the free algebra, had we restricted ourselves to just growth curves for the *regular* representation, our dimension function would also have taken the largest possible value, namely 1. See Section 4.)

Ken Goodearl has raised the interesting question of whether finite GK–dimension always implies that bandwidth dimension is 0. If that were the case, then positive bandwidth dimension might provide a natural extension to GK–dimension by distinguishing among algebras with infinite GK–dimension (that is, taking over where GK gets bad).

Finally a word about our terminology. All rings and algebras are associative with an identity element, and all ring maps preserve the identity. The ground ring for our algebras is a field F. The ring of all $\aleph_0 \times \aleph_0$ column-finite matrices over F, with the rows and columns ordered in the standard way according to ω, is denoted by $M_\omega(F)$. For a subset X of a ring R, the left annihilator of X in R is denoted by $\ell_R(X)$. Similarly $r_R(X)$ denotes the right annihilator.

1 Subalgebras of $B(F)$

In this section we find a family of subalgebras of $B(F)$ associated with growth curves of the form $g(n) = n^r$ where $0 \le r \le 1$.

Recall that a function $g : \mathbf{N} \to \mathbf{R}^+$ is a growth curve for $x \in B(F)$ if for each $n \in \mathbf{N}$ we have $x(n,i) = 0 = x(i,n)$ whenever $i - n > g(n)$. As we observed in the introduction, every matrix in $B(F)$ has a growth curve. We begin by calculating a growth curve for the product of two matrices in $B(F)$.

Lemma 1.1. *Suppose $x, y \in B(F)$ have g and h (respectively) as growth curves. Then a growth curve for the product xy is given by the function $f : \mathbf{N} \to \mathbf{R}^+$ where*

$$f(n) = \max\{g(n) + h(n + [g(n)]), h(n) + g(n + [h(n)])\}$$

and where $[\ \ ]$ denotes the integer part.

Proof. The (n,j) entry of xy is $\sum_{k\ge1} x(n,k)y(k,j)$ and the largest j for which this can be nonzero can't exceed $j = k + [h(k)]$ for $k = n + [g(n)]$. Thus for $j > n + g(n) + h(n + [g(n)])$ the (n,j) entry of xy is zero. Similarly the (i,n) entry of xy is zero whenever $i > n + h(n) + g(n + [h(n)])$. Taking the larger of these two values gives the formula for f. □

If we apply this calculation to growth curves of the form cn^r where $c > 0$ and $0 \le r \le 1$ we can show that the sets $G(r)$ defined in the introduction are in fact subalgebras of $B(F)$.

Proposition 1.2. *Suppose $0 \le r \le 1$. Let*

$$G(r) = \{x \in B(F) : x \text{ has } O(n^r) \text{ growth}\}$$

and for each $c \ge 0$ let

$$W_r(c) = \{x \in B(F) : x \text{ has } cn^r \text{ as a growth curve}\}.$$

Then :

(a) *the* $W_r(c)$ *form a chain of subspaces whose union is* $G(r)$;

(b) *for any* $c_1, c_2 \geq 0$ *we have*

$$W_r(c_1)W_r(c_2) \subseteq W_r(c_3)$$

where $c_3 = \max\left\{c_1 + c_2(1+c_1)^r, c_2 + c_1(1+c_2)^r\right\}$;

(c) $G(r)$ *is a subalgebra of* $B(F)$.

Proof. (a) is trivial and (c) follows immediately from (a) and (b), so we just need to check (b). Let $x \in W_r(c_1)$ and $y \in W_r(c_2)$. We apply Lemma 1.1 with $g(n) = c_1 n^r$ and $h(n) = c_2 n^r$. Since $r \leq 1$ we have

$$\begin{aligned} g(n) + h(n + g(n)) &= c_1 n^r + c_2[n + c_1 n^r]^r \\ &\leq \left(c_1 + c_2(1+c_1)^r\right) n^r \end{aligned}$$

and similarly

$$h(n) + g(n + h(n)) \leq \left(c_2 + c_1(1+c_2)^r\right) n^r$$

as required. □

Remark 1.3. Of course c_3 can be replaced by any larger value in (b) above. In particular

$$c_3 = c_1 + c_2 + c_1 c_2$$

gives a rather simpler value that works for any $r \leq 1$. □

The multiplication law in Proposition 1.2(b) says that the subspaces $W_r(k), k = 1, 2, \ldots$ almost provide a filtering for the subalgebra. In fact when $r = 0$ we really do get a filtering, since in that case Proposition 1.2(b) says that

$$W_0(c_1)W_0(c_2) \subseteq W_0(c_1 + c_2).$$

This filtering will be useful later, so we record some of its properties.

Proposition 1.4. *Let* $W_0(k), k = 1, 2, \ldots$ *be the subspaces of* $G(0)$ *given by Proposition 1.2.*

(a) *For each* k,

$$W_0(k) = \{x \in B(F) : x \text{ has constant bandwidth at most } k\}.$$

(b) $W_0(0) \subseteq W_0(1) \subseteq \ldots$ and $G(0) = \bigcup_{k \geq 1} W_0(k)$. *Also* $W_0(c_1)W_0(c_2) \subseteq W_0(c_1 + c_2)$.

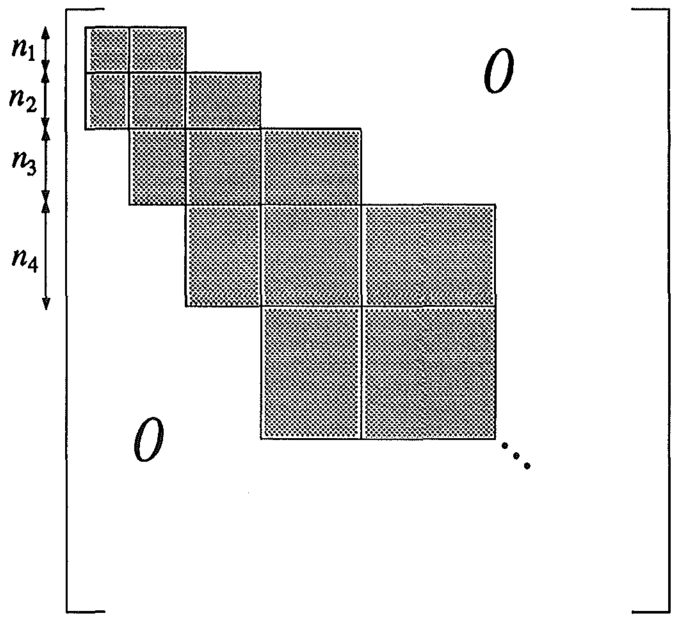
(c) $W_0(0)$ *is a subalgebra of* $G(0)$ *which is isomorphic to* $\prod_{\mathbb{N}} F$.

(d) *For each* k, $W_0(k)$ *is a finitely generated projective right and left* $W_0(0)$*–module.*

Proof. (a), (b), (c) are trivial, and (d) is easy once one splits $W_0(k)$ into a direct sum whose factors correspond to matrices having nonzero entries in exactly one sub-diagonal or super-diagonal.

It may be worth noting here that $G(0)$ has already been studied by Tjukavkin [T], who observed that $G(0)$ is a non-regular ring in which every one-sided ideal is generated by idempotents. It is not hard to see that in fact all the $G(r)$ (where $0 \leq r \leq 1$), and indeed $B(F)$ itself, also have this property.

The other subalgebras $G(r)$ given by Proposition 1.2 also have a filtered structure, but it derives from a block matrix view of growth curves which we shall now describe. We begin with the trivial observation that, just as every $x \in B(F)$ has a growth curve, so every $x \in B(F)$ can be viewed as a block tridiagonal matrix where all the blocks down the main diagonal are square (finite) matrices (see figure).



Of course the sizes of the blocks will vary for different $x \in B(F)$, or even for different $x \in G(r)$, where r is fixed, $0 \leq r \leq 1$. However, once r is fixed, it turns out (see Proposition 1.5 below) that we can choose a fixed "skeleton" of block sizes $n_1, n_2, \ldots$ (see the above diagram), and that we can use these to represent each element of $G(r)$ as a matrix of finite block-bandwidth. (For a matrix in block form, the *block-bandwidth* is just the bandwidth measured in terms of the number of off-diagonal *blocks*, rather than the number of off-diagonal *entries*.) In such a representation an element of $G(r)$ looks like an element of $G(0)$ except that its entries are block matrices. Hence the filtration we have seen on $G(0)$ suggests a way of constructing a filtration on $G(r)$. We set

$$X_r(d) = \{x \in G(r) : x \text{ has block-bandwidth at most } d\} \qquad d = 0, 1, 2, \ldots .$$

It should be noted that the subspaces $X_r(d)$ depend on the choice of skeleton $n_1, n_2, \ldots$ and these values are not unique. For instance, by choosing $n_1 = n_2 = \cdots = n$ we get a series of different filtrations of $G(0)$, one for each value of n.

Proposition 1.5. *For each $r \in [0,1]$ there is a sequence $\{n_k\}$ of block sizes such that the subspaces $X_r(d)$ defined above satisfy*

(a) $X_r(0) \subseteq X_r(1) \subseteq X_r(2) \subseteq \ldots$ *and* $G(r) = \cup_{d\geq 1} X_r(d)$.
Also $X_r(d_1)X_r(d_2) \subseteq X_r(d_1 + d_2)$, *so we have a filtration of* $G(r)$.
(b) $X_r(0)$ *is a subalgebra of* $G(r)$ *isomorphic to* $\prod_{k=1}^{\infty} M_{n_k}(F)$.
(c) *For each d,* $X_r(d)$ *is a finitely generated projective right and left* $X_r(0)$*–module.*

Proof. The details are not difficult, but we omit the proof because we do not use the result later.

2 Linear Growth

Here our aim is to establish the following, somewhat surprising, linear growth result.

Theorem 2.1. *Every countable-dimensional algebra A over a field F has linear growth, that is, A can be embedded in $G(1)$. Thus every countable-dimensional algebra has its bandwidth dimension in* [0,1].

There are two key results which will lead us to the proof of this theorem. The first (Theorem 2.3) calculates the block sizes of a simultaneous block tridiagonal form for a finite number of given linear transformations of a countable-dimensional vector space. The second (Theorem 2.4) is that every countable-dimensional algebra can be embedded in a finitely generated algebra. This was established in 1989 by O'Meara, Vinsonhaler and Wickless. The following elementary lemma is required for Theorem 2.3.

Lemma 2.2. *Let U be a subspace of a vector space V over F, and let $x_1, x_2, \ldots, x_k \in \mathrm{End}_F(V)$. Then*

$$[U : x_1^{-1}(U) \cap x_2^{-1}(U) \cap \ldots \cap x_k^{-1}(U) \cap U] \leq k[V : U].$$

Proof. We have the linear transformation

$$\theta : U \to (V/U) \oplus (V/U) \oplus \cdots \oplus (V/U)$$

from U into k copies of V/U given by $u \to (x_1(u) + U, x_2(u) + U, \ldots, x_k(u) + U)$, whose kernel is $x_1^{-1}(U) \cap \ldots \cap x_k^{-1}(U) \cap U$. Now

$$[U : \ker\theta] \leq \dim\ ((V/U) \oplus \cdots \oplus (V/U)) = k[V : U]$$

and the result follows.

Theorem 2.3. *Let V be any countably-infinite-dimensional vector space over a field F, and let $x_1, \ldots, x_k \in \mathrm{End}_F(V)$. Then there exist finite-dimensional subspaces $U_0 = \{0\}, U_1, U_2, \ldots, U_n, \ldots$ of V such that:*

(1) $V = U_1 \oplus U_2 \oplus \cdots \oplus U_n \oplus \cdots$
(2) $\dim U_n = (2k+1)^{n-1}$ for all $n \geq 1$
(3) $x_i(U_n) \subseteq U_{n-1} \oplus U_n \oplus U_{n+1}$ for $i = 1, \ldots, k$ and $n \geq 1$.

Proof. Let $\{w_1, \ldots, w_n, \ldots\}$ be a fixed basis for V. Set $U_0 = \{0\}$. We shall establish, by induction, the existence of subspaces U_n, V_n of V for $n = 1, 2, \ldots$, with the U_n finite-dimensional, such that the following properties hold for all $n \geq 1$:

(i) $V = U_1 \oplus \cdots \oplus U_n \oplus V_n$
(ii) $V_n = U_{n+1} \oplus V_{n+1}$
(iii) $w_n \in U_1 + \ldots + U_n$
(iv) $x_i(V_{n+1}) \subseteq V_n$ for $i = 1, \ldots, k$
(v) $x_i(U_n) \subseteq U_{n-1} \oplus U_n \oplus U_{n+1}$ for $i = 1, \ldots, k$
(vi) $\dim U_n = 2k(\dim U_1 + \cdots + \dim U_{n-1}) + 1$

For $n = 1$ we simply take $U_1 = < w_1 >$ and let V_1 be any complement of U_1 in V. Now suppose $n \geq 1$ and that we have constructed $U_1, \ldots, U_n, V_1, \ldots, V_n$ satisfying the above properties. The construction of U_{n+1}, V_{n+1} involves several steps.

Firstly, let

$$X = x_1^{-1}(V_n) \cap x_2^{-1}(V_n) \cap \cdots \cap x_k^{-1}(V_n) \cap V_n.$$

By Lemma 2.2 and (i), $[V_n : X] \leq k[V : V_n] = k(\dim U_1 + \cdots + \dim U_n)$. Let $y \in V_n$ be the projection of w_{n+1} on V_n relative to the decomposition $V = (U_1 + \cdots + U_n) \oplus V_n$. Choose a subspace V'_{n+1} of X such that $[X : V'_{n+1}] \leq 1$ and $y \notin V'_{n+1}$. Write

$$V_n = U'_{n+1} \oplus V'_{n+1}$$

for some subspace U'_{n+1} containing y. Note that $w_{n+1} \in U_1 + \cdots + U_n + U'_{n+1}$ and $\dim U'_{n+1} = [V_n : V'_{n+1}] \leq [V_n : X] + 1$. Hence

$$\dim U'_{n+1} \;\leq\; k(\dim U_1 + \cdots + \dim U_n) + 1.$$

For $i = 1, \ldots, k$ we have by induction, using (ii) and (iv), that

$$x_i(U_n) \subseteq x_i(V_{n-1}) \subseteq V_{n-2} = U_{n-1} \oplus V_{n-1} = U_{n-1} \oplus U_n \oplus V_n.$$

Therefore $x_i(U_n)$ is a subspace of $\left(U_{n-1} + U_n + U'_{n+1}\right) \oplus V'_{n+1}$. Let Y_i be the projection of $x_i(U_n)$ on V'_{n+1} relative to this decomposition, and let

$$U''_{n+1} = Y_1 + \cdots + Y_k.$$

Notice that $\dim Y_i \leq \dim x_i(U_n) \leq \dim(U_n)$, whence $\dim U''_{n+1} \leq k(\dim U_n)$. Set

$$U_{n+1} = U'_{n+1} \oplus U''_{n+1},$$

and write

$$V'_{n+1} = U''_{n+1} \oplus V_{n+1}$$

for some subspace V_{n+1}.

Since $V_n = U'_{n+1} \oplus V'_{n+1} = U'_{n+1} \oplus U''_{n+1} \oplus V_{n+1} = U_{n+1} \oplus V_{n+1}$, we have (ii), and hence also (i) for $n+1$. From $w_{n+1} \in U_1 + \cdots + U_n + U'_{n+1} \subseteq U_1 + \cdots + U_n + U_{n+1}$, we get (iii). Property (iv) follows from $x_i(V_{n+1}) \subseteq x_i(X) \subseteq V_n$. Also $x_i(U_n) \subseteq \left(U_{n-1} \oplus U_n \oplus U'_{n+1}\right) \oplus U''_{n+1} = U_{n-1} \oplus U_n \oplus U_{n+1}$ gives (v). For (vi), observe that

$$\begin{aligned} \dim U_{n+1} &= \dim U'_{n+1} + \dim U''_{n+1} \\ &\leq (k(\dim U_1 + \cdots + \dim U_n) + 1) + k(\dim U_n) \\ &\leq 2k(\dim U_1 + \cdots + \dim U_n) + 1. \end{aligned}$$

By expanding U_{n+1} to include more of V_{n+1}, we can arrange our choice of U_{n+1} and V_{n+1} such that in addition to properties (i), (ii), ..., (v), we have

$$\dim U_{n+1} = 2k(\dim U_1 + \cdots + \dim U_n) + 1.$$

This completes the induction.

Property (1) is an immediate consequence of (i) and (iii). The recursive relation (vi), together with $\dim U_1 = 1$, yields (2). Finally (3) is just (v). $\square$

Theorem 2.4. *Every countable-dimensional algebra A over a field can be embedded in some finitely generated algebra (in a 2–generator algebra in fact).*

Proof. [OVW] established the corresponding result for countable rings but, as pointed out in the Introduction to [OVW], the same techniques also work for algebras. Specifically, using the notation of the proof of the Theorem in [OVW], the modifications required are as follows: Suppose R is a countable-dimensional algebra. Let $\{r_1, r_2, \ldots, r_n, \ldots\}$ be a basis for R. Define the maps

$$R \xrightarrow{\phi_1} eSe \xrightarrow{\phi_2} fTf \xrightarrow{\phi_3} T$$

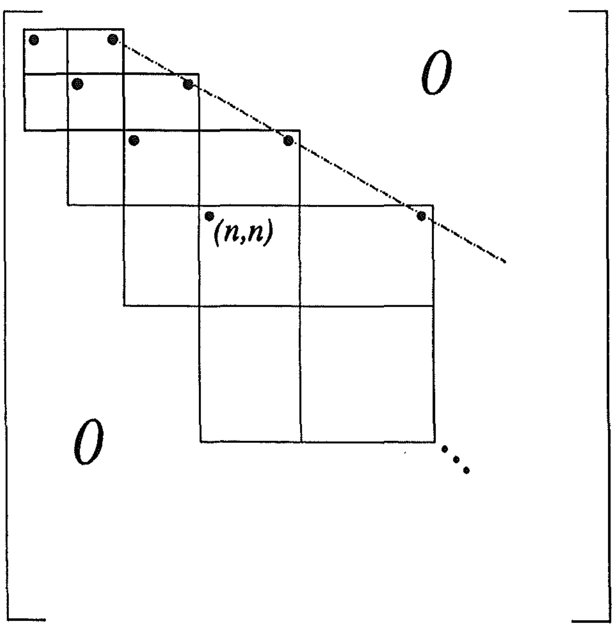
exactly as before, and note that these are now algebra maps. Hence $\theta = \phi_3\phi_2\phi_1 : R \to T$ is an algebra embedding whose image is contained in the subalgebra generated by $\phi(a), \phi(b), f, v, w$.

Proof of Theorem 2.1. By Theorem 2.4 we can reduce to the case where A is a subalgebra of $Q = M_\omega(F)$ generated by a finite number of elements, say $x_1, x_2, \ldots, x_k$ (we could even take $k = 2$). By Theorem 2.3, there is a similarity transformation of Q under which all the x_i are simultaneously in block tridiagonal form (see figure below) and where the sizes of the diagonal blocks are

$$1,\ 2k+1,\ (2k+1)^2, \ldots, (2k+1)^n, \ldots$$

Consider the dotted piecewise linear curve obtained by joining the outside corners of the upper blocks. This is clearly an upper growth curve for all the x_i. Viewed as a growth curve, it has the equation

$$g(n) \;=\; 4k(k+1)n - \left(4k^2 + 2k - 1\right)$$



because at the (n,n) position for any n of the form

$$\begin{aligned} n &= \text{row index of the first entry of the } m^{\text{th}} \text{ diagonal block} \\ &= \text{sum of the first } m-1 \text{ block sizes } +1 \\ &= 1+(2k+1)+(2k+1)^2+\cdots+(2k+1)^{m-2}+1 \\ &= \frac{(2k+1)^{m-1}-1}{2k}+1, \end{aligned}$$

the bandwidth of the tridiagonal block structure is

$$(2k+1)^{m-1}+(2k+1)^m-1 \;=\; 4k(k+1)n-\left(4k^2+2k-1\right) \;=\; g(n).$$

Hence the growth curve is a straight line! Clearly $g(n)$ is also a lower growth curve. This shows that the similarity transformation puts all the x_i in $G(1)$. Therefore since $x_1,\ldots,x_k$ generate A, and $G(1)$ is a subalgebra of $M_\omega(F)$ by Proposition 1.2, the image of A lies inside $G(1)$ as well.

It is worth noting that the proof of Theorem 2.1 shows that linear growth for any finitely generated subalgebra A of $M_\omega(F)$ can be demonstrated using just a similarity transformation. We shall see examples later (Theorem 4.2 and Example 4.4) of finitely generated subalgebras of $M_\omega(F)$ which turn out to have sublinear growth (even embed in $G(0)$) but for which no similarity transformation can embed them in a $G(r)$ for any $r<1$.

3 Bandwidth dimension of purely infinite algebras

In this section we show that, in general, we cannot achieve sublinear growth for countable-dimensional algebras (Theorem 3.3). Thus the embedding found in Theorem 2.1 is the best possible.

The idea is to view elements of $B(F)$ as block tridiagonal matrices acting on a vector space decomposition

$$V=U_1\oplus U_2\oplus\cdots\oplus U_n\oplus\cdots$$

as in the proof of Theorem 2.3. The following lemma gives us a condition which ensures that the subspaces grow too quickly for sublinear growth.

Lemma 3.1. *Let $x,z\in \mathrm{End}_F(V)$ be such that x,z are both one-to-one, but $Im\ x\cap Im\ z=0$. Suppose there are finite-dimensional subspaces $U_1,U_2,\ldots$ of V such that $V=\oplus_{i\geq 1}U_i$ and such that for both the maps $f=x$ and z we have*

$$f(\oplus_{i=1}^{k}U_i)\;\subseteq\;\oplus_{i=1}^{k+1}U_i \tag{$*$}$$

for each $k \geq 1$. Then for each $k \geq 2$ we have

$$\dim U_k \ \geq \ \sum_{i=1}^{k-1} \dim U_i.$$

Proof. For each k let $W_k = \oplus_{i=1}^{k} U_i$ so that by $(*)$ we have xW_{k-1}, $zW_{k-1} \subseteq W_k$. Also $xW_{k-1} \cap zW_{k-1} \subseteq Im\ x \cap Im\ z = 0$. Since x, z are both one-to-one this implies

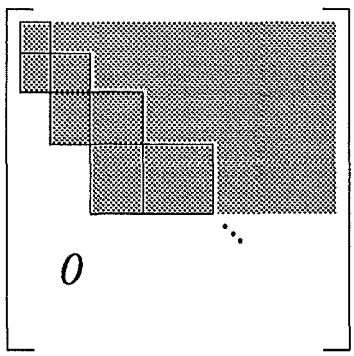
$$\begin{aligned} \dim W_k &\geq \dim(xW_{k-1} + zW_{k-1}) \\ &= \dim(xW_{k-1}) + \dim(zW_{k-1}) \\ &= \dim W_{k-1} + \dim W_{k-1} \end{aligned}$$

and so

$$\begin{aligned} \dim U_k &= \dim W_k - \dim W_{k-1} \\ &\geq \dim W_{k-1} \end{aligned}$$

as desired.

The hypothesis $(*)$ says that x, z are simultaneously in "block upper Hessenberg" form (see figure below).

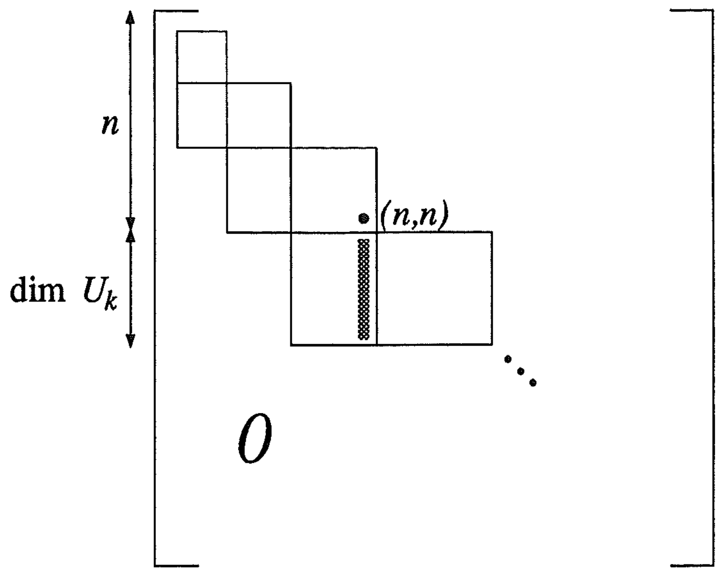


This form is possible for any finite collection of column-finite matrices, as long as the block sizes are chosen properly. It determines a lower growth curve for x and z.

Lemma 3.1 gives an estimate for the bandwidth of x and z (at least below the main diagonal). In the notation of the proof let

$$n = \sum_{i=1}^{k-1} \dim U_i.$$

Then we can estimate how far down the n^{th} column we need to go before all the entries for x and z become zero. Indeed, in the nth column the entries are zero beyond the $(n + \dim U_k)$th row. Lemma 3.1 says that $\dim U_k \geq n$.



If the sub-diagonal blocks are chosen to be as short as possible, then there must be a nonzero entry in the final row of each of these blocks. If we are looking for the smallest possible bandwidth, then the "best" place for such a nonzero entry is the bottom righthand corner of each block (this gives the smallest deviation from the main diagonal). For the (n, n) entry in the diagram we would thus have a bandwidth of $\dim U_k \geq n$. Hence if $g(n)$ is a growth curve for x and z, there are infinitely many n for which $g(n) \geq n$. Therefore any common growth curve for the maps x, z in Lemma 3.1 must be at least linear. All that remains is to find a way of ensuring that any (faithful) representation in $\mathrm{End}_F(V)$ of a suitable algebra A always contains such a pair of maps.

Proposition 3.2. *Let A be a countable-dimensional algebra over F containing elements w, x, y, z satisfying*

$$yx = 1 = wz \quad \text{and} \quad yz = 0.$$

Then any copy of A inside $B(F)$ has at least linear growth, and so A has bandwidth dimension one.

Proof. The same equations will be satisfied by the images $\bar{w}, \bar{x}, \bar{y}, \bar{z}$ of w, x, y, z in $B(F)$. The first two equations force $\bar{x}$ and $\bar{z}$ to be one-to-one, and if $\bar{x}(v_1) = \bar{z}(v_2) \in Im\ \bar{x} \cap Im\ \bar{z}$ then $v_1 = \bar{y}\bar{x}(v_1) = \bar{y}\bar{z}(v_2) = 0$ and so $Im\ \bar{x} \cap Im\ \bar{z} = 0$. So the result follows from Lemma 3.1 and the above discussion.

It is easy to construct matrices w, x, y, z in $B(F)$ which satisfy the hypotheses of Proposition 3.2. More generally, these hypotheses are satisfied by two fairly large classes of algebras. One of these classes needs a name:

Definition. *We say that a ring R* is ***purely infinite*** *if $R \cong R \oplus R$ as right R–modules.*

If R is also regular and right self-injective, then this usage agrees with that in [G3, pp116–117].

Theorem 3.3. *Let A be a countable-dimensional algebra over F such that either*

(i) *A is purely infinite, or*

(ii) *A is regular and $A \oplus A \lesssim A$ as right A–modules.*

Then every copy of A in $B(F)$ has at least linear growth, and so A has bandwidth dimension one.

Proof. In either case A contains a pair of orthogonal idempotents e, f such that $eA \cong A_A \cong fA$. By [Jac 2, Proposition 4, page 51] there are elements $w, x, y, z \in A$ such that $yx = 1$, $xy = e$ and $wz = 1$, $zw = f$. But then $0 = ef = xyzw$ and so $yz = 0$. The theorem thus follows from Proposition 3.2.

If A is not regular then we cannot weaken condition (i) to say $A \oplus A \lesssim A$. For example, consider the free algebra A on two generators. Then $A \oplus A \lesssim A$, because A is not a right Ore domain, but the conditions of Proposition 3.2 are not satisfied (since A is a domain, $yz = 0$ forces $y = 0$ or $z = 0$). And in any event we shall see in the next section that A can in fact be embedded in $G(0)$!

4 Bandwidth dimension of free algebras

The main result of this section is that the free algebra on any finite or countably infinite number of generators can be embedded in $G(0)$ and so has bandwidth dimension 0 (Theorem 4.2). This result is rather surprising since the free algebra is often thought of as being "large": its GK–dimension is ∞, for example. At the end of this section we shall look in more detail at the differences between GK–dimension and bandwidth dimension.

The following lemma contains the key idea for representing the free algebra in terms of matrices with small bandwidth. We recall the notation e_{ij} for the matrix unit with all entries zero except for a one in the (i, j) position.

Lemma 4.1 *Let F be any field (or indeed any ring with identity) and let $F\{x, y\}$ be the free algebra in two indeterminates x, y. Suppose $w = a_1 a_2 \ldots a_n$ is any word in x, y of degree n. Then there is an algebra homomorphism $\varphi : F\{x, y\} \to M_{n+1}(F)$ such that:*

(i) *the only nonzero entries in* $\varphi(x)$ and $\varphi(y)$ *are on the superdiagonal, and these are all* 1's ;

(ii) $\varphi(w) = e_{1,n+1}$;

(iii) if v *is any other word in* x, y *having degree* n *then* $\varphi(v) = 0$;

(iv) if $\rho \in F\{x,y\}$ *then the* $(1, n+1)$ *entry of* $\varphi(\rho)$ *is the coefficient of* w *in* ρ.

Proof. To construct φ it is enough to specify the images of x, y. To do this, partition the set $\{1, 2, \ldots, n\}$ as $X \cup Y$ where $i \in X \Leftrightarrow a_i = x$ and $i \in Y \Leftrightarrow a_i = y$. Set

$$\varphi(x) = \sum\nolimits_{i \in X} e_{i,i+1} \text{ and } \varphi(y) = \sum\nolimits_{i \in Y} e_{i,i+1}.$$

Thus (i) is certainly true.

Let N be the set of strictly upper triangular matrices in $M_{n+1}(F)$. Then $\varphi(x), \varphi(y) \in N$ and $N^n \subseteq Fe_{1,n+1}$. Hence if $v = b_1 b_2 \ldots b_n$ is any word of degree n in x, y then the matrix $\varphi(v)$ has zero entries except in the $(1, n+1)$ position. This $(1, n+1)$ entry is given by

$$\sum\nolimits_{i_k} \overline{b}_1(1, i_1)\overline{b}_2(i_1, i_2) \ldots \overline{b}_n(i_{n-1}, n+1)$$

where each $\overline{b}_k = \varphi(b_k)$. But as each $\overline{b}_i$ is $\varphi(x)$ or $\varphi(y)$, the terms in the above sum can be nonzero only if $i_1 = 2, i_2 = 3, \ldots, i_{n-1} = n$ (because of (i)). So the $(1, n+1)$ entry of $\varphi(v)$ is simply

$$\overline{b}_1(1,2)\overline{b}_2(2,3) \ldots \overline{b}_n(n, n+1)$$

and this term can be nonzero only if $\overline{b}_i(i, i+1) \neq 0$ for each i. By the definition of $\varphi(x)$ and $\varphi(y)$ we thus have $\overline{b}_i = \varphi(x)$ when $i \in X$ and $\overline{b}_i = \varphi(y)$ when $i \in Y$. Hence the $(1, n+1)$ entry is nonzero only when $v = w$ and then the entry is simply 1. Thus (ii) and (iii) are proven.

Since $N^{n+1} = 0$ any word of higher degree is mapped to zero by φ. Finally since $\varphi(x), \varphi(y)$ are zero except on the superdiagonal, no word of smaller degree than n can yield a nonzero $(1, n+1)$ entry. Hence (iv) is true too.

Theorem 4.2. *Let* F *be a field and let* $F\{x,y\}$ *be the free algebra in two (non-commuting) indeterminates. Then* $F\{x,y\}$ *can be embedded as a subalgebra of* $G(0)$. *In particular* $F\{x,y\}$ *has bandwidth dimension* 0.

Proof. It is enough to embed $F\{x,y\}$ as a subalgebra of a direct product $\prod_{k=1}^{\infty} M_{n_k}(F)$ in such a way that each component of the images of x, y is a matrix whose only nonzero entries lie on the superdiagonal.

Let $w_1, w_2, \ldots, w_k, \ldots$ be a list of all the words in x, y (i.e. monomials in $F\{x,y\}$). This is possible because there are only countably many such words. For each k, let d_k be the degree of w_k and let $n_k = d_k + 1$. Let $\varphi_k : F\{x, y\} \to M_{n_k}(F)$ be the algebra homomorphism given by Lemma 4.1 for the word $w = w_k$. Let $\varphi : F\{x, y\} \to \prod_k M_{n_k}(F)$ be the algebra homomorphism determined by the φ_k. Then Lemma 4.1 (iv) ensures that φ is an embedding, and Lemma 4.1 (i) ensures that the images of x, y have the required form.

Remarks. (1) In terms of a concrete realization of $F\{x, y\}$ inside $G(0)$, the proof of Theorem 4.2 shows that we can choose x and y as block diagonal $\omega \times \omega$ matrices where the blocks are finite and matching, and constitute all pairs of finite matrices of the following form: the first is an $n \times n$ matrix whose only nonzero entries lie on the superdiagonal and are all 1, while the second is an $n \times n$ matrix of the same form but with the 1's in complementary positions on the superdiagonal. For example, one such matching pair of 5×5 blocks is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} .$$

(2) Note that Theorem 4.2 also holds for a free algebra on any finite or countably infinite number of indeterminates because such algebras can be embedded in $F\{x, y\}$.

In terms of bandwidth dimension then, the free algebra $F\{x, y\}$ is "small". On the other hand we saw in Theorem 3.3 that any purely infinite algebra A has bandwidth dimension one and so, presumably, is "large". In this context it may be worth pointing out the following, probably very well-known, result.

Proposition 4.3. *If A is a purely infinite algebra over F, then A contains a subalgebra isomorphic to the free algebra on two generators.*

Proof. Since $A \cong A \oplus A$ as right A–modules, we have seen that A contains elements w, x, y, z such that $yx = 1 = wz$ and $yz = 0$. But then $r_A(x) = 0 = r_A(z)$ and $xA \cap zA = 0$. Now by the standard argument used in [Jat, p45] for non-Ore domains, it can be seen that x, z generate a copy of the free algebra (with identity) inside A.

Thus the free algebra on two generators is certainly "smaller" than any purely infinite algebra, and Theorem 4.2 quantifies the difference. On the other hand the free algebra has infinite GK–dimension, so this may be a good place to examine more closely the similarities and differences between bandwidth dimension and GK–dimension.

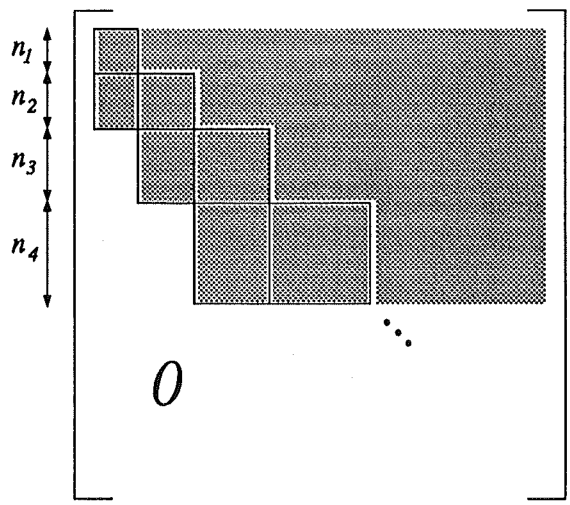
Recall firstly how the GK–dimension of a finitely generated F–algebra A is calculated (see [KL]). We begin with a finite dimensional subspace U of A which contains 1 and generates A

as an F–algebra. Then the GK–dimension of A is given by

$$GK-\dim A \;=\; \limsup_k \Big(\frac{\log(\dim U^k)}{\log k}\Big)$$

where, as usual, U^k is the subspace of A generated by all products of k elements of U.

We can interpret this calculation in terms of the "block upper Hessenberg" form which we observed after Lemma 3.1, and so use it to find a "lower growth curve" for the generators of A. Consider the regular representation of A where A acts via left multiplication on itself. Then $U \subseteq U^2 \subseteq U^3 \subseteq \ldots$ is an increasing chain of subspaces of A whose union is A, and for any $u \in U$ we clearly have $uU^k \subseteq U^{k+1}$. Hence these subspaces let us put all the elements of U simultaneously into block upper Hessenberg form (see figure below)



where the block sizes shown are determined by the condition

$$\dim U^k \;=\; n_1 + n_2 + \cdots + n_k.$$

In terms of this representation, the above calculation of $GK-\dim A$ essentially seeks to express $\dim U^k$ (= the total size of the first k diagonal blocks) as a polynomial in k (= the number of blocks). This is because $\dim U^k$ "looks increasingly like" k^t if $GK-\dim A = t$. In this sense GK–dimension measures how far we need to go down each column before all the entries are zero in the regular representation of the generators of A. Thus loosely speaking, we can view $GK-\dim A$ as determining a "lower growth curve" for the generators of A. This view of GK–dimension allows us to draw some comparisons with bandwidth dimension.

1. The GK–dimension of A tells us about a *lower* growth curve for the generators of A, whereas the bandwidth dimension of A gives us an *upper and lower* growth curve for these generators. (This might lead us to expect the bandwidth dimension to be larger than the GK–dimension.)

2. The lower growth curves associated with GK–dimension are relative to the regular representation of A and, indeed, are relative to bases of A which correspond to ascending subspaces

$$U \subseteq U^2 \subseteq U^3 \subseteq \dots$$

where U generates A as an F–algebra. On the other hand the growth curves determined by the bandwidth dimension are the "best possible" among all possible (faithful) matrix representations. (This factor would tend to make the bandwidth dimension smaller than the GK–dimension.)
3. GK–dimension measures the growth by comparing the total size of the first k blocks with k, but the bandwidth dimension measures the growth by comparing the size of the next block with the total size of all the preceding blocks. (This probably just results in a different scale being used for the two dimensions.)

Because of (1) and (3) it is difficult to make more precise comparisons between the two dimensions: the regular representation of A need not give rise to row-finite matrices, and so there may not be any upper growth curves at all. For some algebras, however, we do get an upper growth curve "free of charge" and we can compare the actual calculation of the two dimensions. In the following two examples there is a close connection between the number k of diagonal blocks (at a given stage) and their total size. With this type of example, we can use the lower growth curve on its own to estimate the GK–dimension.

Example 4.4. Let A be the free algebra $F\{x, y\}$ on two generators. For each integer $k \geq 0$ let U_k be the subspace of A spanned by all monomials of degree exactly k, and notice that $\dim U_k = 2^k$.

To calculate $GK-\dim A$ we use $U = U_0 \oplus U_1$ and find that

$$\begin{aligned}\dim U^k &= \dim(U_0 \oplus U_1 \oplus \dots \oplus U_k) \\ &= 1 + 2 + 2^2 + \dots + 2^k \\ &= 2^{k+1} - 1\end{aligned}$$

and this gives exponential growth when compared with the number of blocks k. Thus $GK-\dim A = \infty$.

If we represent the generators x, y relative to a basis which corresponds to the decomposition $A = U_0 \oplus U_1 \oplus \dots$ then we get lower triangular matrices (since $xU_k \subseteq U_{k+1}$ and $yU_k \subseteq U_{k+1}$) so the lower growth curve will also be an upper growth curve. If we compare the size of the $(k+1)$st block (namely, 2^{k+1}) with the total size of the preceding k blocks (namely, $1 + 2 + \dots + 2^k = 2^{k+1} - 1$) we see that the growth curve is linear. Indeed, it wouldn't make any difference what basis we chose for A here: the actions of x, y on A make x, y one-to-one maps such that $Im\ x \cap Im\ y = 0$ and so Lemma 3.1 shows that we must get at least linear growth. That is, if we restricted ourselves to the regular representation, the bandwidth dimension of the free algebra A would take the largest possible value, 1. But by allowing ourselves to choose from all possible matrix representations we can find a much slimmer representation, as Theorem 4.2 shows.

Example 4.5. Let A be the polynomial algebra $F[x, y]$ in two (commuting) indeterminants. Once again, for each $k \geq 0$, let U_k be the subspace generated by all monomials of degree exactly k. This time, $\dim U_k = k + 1$ but otherwise the calculations are similar to those in Example 4.4. Thus we calculate $GK-\dim A$ using $U = U_0 \oplus U_1$ and find that

$$\dim U^k = \tfrac{1}{2}(k+1)(k+2).$$

This time we have quadratic growth and $GK-\dim A = 2$. Once again the lower growth curve is also an upper growth curve, and the size of the $(k+1)$st block is $k+2$ whereas the total size of the preceding blocks is $\frac{1}{2}(k+1)(k+2)$. This gives a growth curve of the form $ck^{1/2}$ and so we would get a bandwidth dimension of $\frac{1}{2}$ if we restricted ourselves to the regular representation and to this type of basis.

A similar calculation with $A = F[x_1, \ldots, x_n]$ would give $GK-\dim A = n$ and the growth curve for A would suggest a bandwidth dimension of $1 - \frac{1}{n}$. In fact it is not difficult to show that $F[x_1, \ldots, x_n]$ embeds in $G(0)$, so its bandwidth dimension is 0.

As with GK–dimension, bandwidth dimension behaves nicely for subalgebras, finite subdirect products and finite matrix rings. Namely

1. if the algebra A embeds in the algebra B, then the bandwidth dimension of A is at most the bandwidth dimension of B;
2. the bandwidth dimension of a finite subdirect product cannot exceed that of its factors;
3. for any algebra A and positive integer n, the algebras A and $M_n(A)$ have the same bandwidth dimension.

On the other hand, GK–dimension has the very useful property that if I is an ideal of the algebra A, then $GK-\dim A/I \leq GK-\dim A$. But this property fails very badly for bandwidth dimension. For example, take A to be the free algebra on a countably infinite set and choose I so that A/I is purely infinite. Then A has bandwidth dimension 0 but A/I has bandwidth dimension 1 (Theorems 3.3 and 4.2).

5 Regular self-injective subrings of $G(0)$

As a further illustration of how a sublinear growth condition placed on a ring can be reflected in a purely ring-theoretic property, we present the following result.

Theorem 5.1. *Any regular right self-injective ring R which embeds in $G(0)$ must have bounded index of nilpotence.*

For the proof we need to view $G(0)$, the algebra of finite bandwidth matrices, as a filtered algebra with certain properties (see section 1).

Lemma 5.2. *Let $G(0) = \cup_{k=0}^{\infty} W_0(k)$ be the filtering of $G(0)$ into the subspaces*

$$W_0(k) = \{x \in G(0) \mid x \text{ has bandwidth at most } k\}.$$

Then each $W_0(k)$ has a bound ($2k+1$ in fact) on the number of independent nonzero pairwise isomorphic $W_0(0)$ submodules.

Proof. For notational convenience, we drop the subscript 0 in $W_0(k)$. Note that $W(0)$ is a regular ring with bounded index of nilpotence. By Proposition 1.4(d), $W(k)$ is a finitely generated projective module over $W(0)$. The Lemma now follows from [G3, Corollaries 7.3, 7.13].

Proof of Theorem 5.1. We first make the following claim.

Claim: If e is an idempotent of R such that eRe has unbounded index of nilpotence, then $eRe \not\subseteq W(k)$ for any positive integer k.

For suppose $eRe \subseteq W(k)$ for some k. By Lemma 5.2 there is some bound, t say, on the number of independent nonzero isomorphic $W(0)$–submodules of $W(k)$. Since eRe has index greater than t, [G3, Corollary 7.3] shows that eRe contains a direct sum $X_1 \oplus \cdots \oplus X_{t+1}$ of nonzero pairwise isomorphic cyclic left eRe–modules. But now the X_i generate independent isomorphic left $W(0)$–submodules of $W(k)$, a contradiction. Thus the claim is true.

Now suppose R has unbounded index of nilpotence. Then we can find an infinite set $\{g_n\}_1^{\infty}$ of orthogonal idempotents of R such that each g_nRg_n has unbounded index. (See [G3]. For example if R is directly infinite we can use [G3, 5.6 and 7.3]. For the Type II_f case, use 10.28 and 7.17, while for Type I_f use 10.24. Now combine these by the Type decomposition in 10.22.) Each $g_n \in W(k_n)$ for some positive integer k_n because $R \subseteq G(0) = \cup W(k)$. Furthermore we can arrange for the k_n to form a strictly increasing sequence. From above $g_nRg_n \not\subseteq W(2k_n)$, whence for each n we can choose $x_n \in Rg_n$ such that $x_n \notin W(2k_n)$. By right self-injectivity of R, there is an $x \in R$ with $xg_n = x_n$ for all n (this is the critical use of injectivity). But $x \in W(m)$ for some m, and so from Proposition 1.4 we have that $x_n = xg_n$ implies $x_n \in W(m+k_n)$, which for large n contradicts $x_n \notin W(2k_n)$. It must therefore be that

R has bounded index.

Corollary 5.3. *Let $\{n_k\}$ be any unbounded sequence of positive integers and let*

$$A = \prod_{k=1}^{\infty} M_{n_k}(F).$$

Then A does not have constant $(O(1))$ growth. However the bandwidth dimension of A is 0.

Proof. A is a regular right self-injective ring of unbounded index, hence A does not embed in $G(0)$ by Theorem 5.1. Let r be any positive real number. We can embed A in $B(F)$ so that n^r is a growth curve for A, simply by padding out the usual block diagonal representation of A, and repeating each block often enough until the increasing curve n^r has allowed a bandwidth large enough to accommodate the next block. Hence

$$\inf\{r \mid A \text{ embeds in } G(r)\} = 0$$

which says that the bandwidth dimension of A is 0.

It is certainly not the case that all regular subrings of $G(0)$ have bounded index (for example, consider the subalgebra of $G(0)$ consisting of all $\omega \times \omega$ matrices with an arbitrary finite block in the top left corner and scalars down the diagonal). In fact regular subrings of $G(0)$ need not even have all their primitive factors artinian. For example we can obtain a copy of $\varinjlim M_{2^n}(F)$ in $G(0)$ by considering all matrices of the form

$$\begin{bmatrix} B & & & O \\ & B & & \\ & & B & \\ O & & & \ddots \end{bmatrix}$$

where $B \in M_{2^n}(F)$ for some n (this is a simple regular algebra which is not artinian [G3, Example 8.1]).

Theorem 5.1 suggests a list of interesting questions for regular rings:

(1) Must a regular subring of $G(0)$ be directly finite?

(2) Must a directly finite regular subring of $G(0)$ be unit-regular?

(3) If a regular right self-injective algebra has bandwidth dimension 0, must it be of Type I?

(4) Can the free regular algebra on a countable set be embedded in $G(0)$? (Since the free regular algebra embeds in $\prod_{n=1}^{\infty} M_n(F)$ (see [GMM]), it at least has bandwidth dimension 0 by Corollary 5.3.)

By [GMM] the free regular algebra is directly finite but not unit-regular, and so a positive answer to (4) would imply a negative answer to (2). However we suspect that of this list, it is (4) that has the negative answer.

6 Algebras not embeddable in G(0)

We have seen that all countable-dimensional algebras fit in $G(1)$, that the free algebra fits in $G(0)$, and that purely infinite algebras do not fit in $G(r)$ for any $r < 1$ (see Theorems 2.1, 4.2 and 3.3). We also know of uncountable-dimensional algebras which do fit in the middle (Corollary 5.3). Are there countable-dimensional algebras which also fit in the middle? In this section we construct for each $0 < r \leq 1$ a finitely generated subalgebra of $G(r)$ which cannot be embedded in $G(0)$.

The following lemma will help us construct some transformations in $\mathrm{End}_F(V)$ which cannot be represented by a matrix in $G(0)$.

Lemma 6.1. *Let V be a countably-infinite dimensional vector space over F and suppose $x \in \mathrm{End}_F(V)$ has a matrix with finite bandwidth n relative to some basis of V.*

(a) *If* $\ker x = 0$*, then $Im\ x$ has codimension at most n in V.*

(b) *If $Im\ x = V$, then* $\ker x$ *has dimension at most n.*

Proof. We can represent x by a block tridiagonal matrix where all the blocks are $n \times n$ matrices. This means we have a sequence $U_1, U_2, \ldots$ of subspaces of V such that $V = \bigoplus_k U_k$, $\dim U_k = n$ for each k, and

$$xU_1 \subseteq U_1 \oplus U_2$$
$$xU_k \subseteq U_{k-1} \oplus U_k \oplus U_{k+1} \quad (k > 1).$$

Suppose firstly that $\ker x = 0$. To prove (a) consider any finite-dimensional subspace W of V such that $W \cap Im\ x = 0$. It is enough to show that $\dim W \leq n$. By choosing k large enough we can assume that $W \subseteq U_1 \oplus \cdots \oplus U_k$. Now $x(U_1 \oplus \cdots \oplus U_{k-1}) \subseteq U_1 \oplus \cdots \oplus U_k$ and $W \cap x(U_1 \oplus \cdots \oplus U_{k-1}) = 0$. So comparing dimensions gives

$$\begin{aligned} kn &= \dim(U_1 \oplus \cdots \oplus U_k) \\ &\geq \dim W + \dim(x(U_1 \oplus \cdots \oplus U_{k-1})) \\ &= \dim W + \dim(U_1 \oplus \cdots \oplus U_{k-1}) \qquad \text{since } \ker x = 0 \\ &= \dim W + (k-1)n \end{aligned}$$

and the result follows.

To prove (b) we essentially use a dual argument. Fix k and consider $W = (U_1 \oplus \ldots \oplus U_k) \cap \ker x$. It is enough to show that $\dim W \leq n$ (independent of k). Let $\pi : V \to U_1 \oplus \ldots \oplus U_{k-1}$ be the projection map determined by $V = \bigoplus_i U_i$. Then

$$\pi x \left(\bigoplus_{k+1}^{\infty} U_i \right) \subseteq \pi \left(\bigoplus_{k}^{\infty} U_i \right) = 0$$

and so

$$\bigoplus_{i=1}^{k-1} U_i = \pi V = \pi x V = \pi x \left(\bigoplus_{i=1}^{k} U_i \right).$$

Now looking at the dimensions gives

$$\begin{aligned}\dim \left(\bigoplus_{i=1}^{k} U_i \right) &= \dim \left(\bigoplus_{i=1}^{k-1} U_i \right) + \dim \left(\ker (\pi x) \cap \bigoplus_{i=1}^{k} U_i \right) \\ &\geq \dim \left(\bigoplus_{i=1}^{k-1} U_i \right) + \dim W\end{aligned}$$

and so $\dim W \leq n$ as required.

Remark 6.2. We saw at the end of Section 2 that not only do finitely generated subalgebras of $M_\omega(F)$ embed in $G(1)$, they are in fact similar to subalgebras of $G(1)$. Every one-generator algebra A can be embedded as a subalgebra of $G(0)$ (and so has bandwidth dimension zero). To see this, notice that A is either finite dimensional (in which case the result is trivial) or else isomorphic to the polynomial algebra $F[x]$, and in that case we can map the generator to the standard shift matrix

$$\begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \\ 0 & 1 & 0 & \\ & \ddots & \ddots & \end{bmatrix}$$

However Lemma 6.1 implies that there are one-generator subalgebras A of $M_\omega(F)$ which are not similar to any subalgebra of $G(0)$. Indeed we can let A be the subalgebra generated by any matrix x corresponding to a one-to-one linear transformation whose range has infinite codimension.

Theorem 6.3. *Suppose $0 < r \leq 1$. Then there is a four generator subalgebra A of $G(r)$ such that A cannot be embedded in $G(0)$.*

Proof. Let V be a countable-dimensional vector space over F with basis $\{v_n : n \in \mathrm{N}\}$ and let $Q = \mathrm{End}_F(V)$. Let $f : \mathrm{N} \to \mathrm{N}$ be the function given by

$$f(n) = n + [n^r]$$

where $[a]$ is the greatest integer less than or equal to a. Let A be the subalgebra of Q generated by w, x, y, z where

$$x(v_n) = v_{n+1} \qquad \text{for all } n \geq 1$$

$$y(v_n) = \begin{cases} 0 & \text{if } n = 1 \\ v_{n-1} & \text{if } n > 1 \end{cases}$$

$$z(v_n) = v_{f(n)} \qquad \text{for all } n \geq 1$$

$$w(v_n) = \begin{cases} 0 & \text{if } n \notin Imf \\ v_k & \text{if } n = f(k). \end{cases}$$

Notice that w is well-defined since f is a strictly increasing function (and so one-to-one). If we represent elements of Q in terms of the given basis then w, x, y, z all belong to $G(r)$, and so A is a subalgebra of $G(r)$.

For each integer $n \geq 1$ let $e_n = x^{n-1}y^{n-1} - x^n y^n$. We now assemble a list of relations which are satisfied by w, x, y, z and the e_n, the idea being that these same relations have to be satisfied in any other copy of A in Q. Firstly it is easy to check that

(i) $yx = 1 = wz$, and

(ii) $e_1 = 1 - xy \neq 0$.

Next, by [Jac 2, Proposition 4, page 51] we see that

(iii) the e_n are nonzero, orthogonal, pairwise equivalent idempotents.

Notice also that for each n, the transformation e_n is just the natural projection of V onto the subspace spanned by v_n. Hence

(iv) $we_n = 0 = e_n z$ for all $n \notin Imf$. Notice here that, since the function f is of the form $f(n) = n + g(n)$ where g is an unbounded, increasing function, there are infinitely many $n \notin Imf$. Indeed each time g is increases in value, say $g(n) < g(n+1)$, the function f skips a value in N since

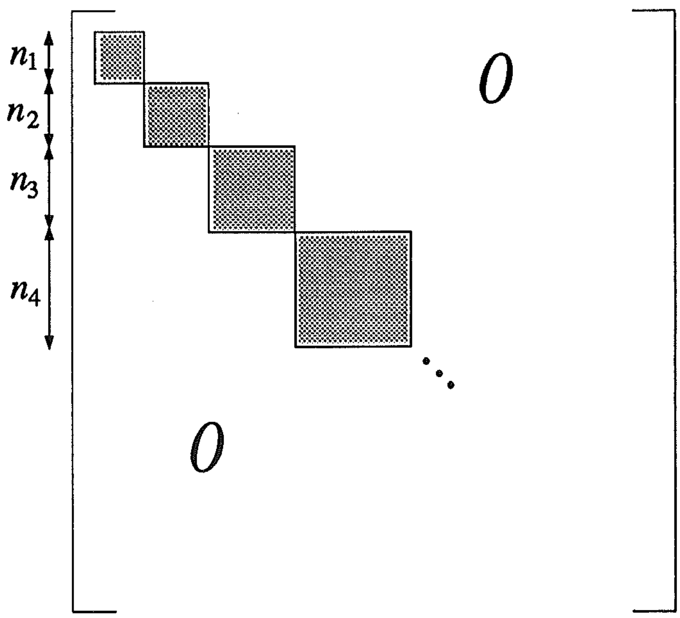
$$f(n) = n + g(n) < n + g(n+1) < (n+1) + g(n+1) = f(n+1).$$

Now any embedding of A in $G(0)$ will produce elements w, x, y, z and $\{e_n\}$ in $G(0)$ which satisfy the relations (i) — (iv) above. By (i) the map w would then be onto, and by (ii), (iii) and (iv) w would have an infinite-dimensional kernel. But by Lemma 6.1 such a map cannot be in $G(0)$, and so A cannot be embedded in $G(0)$.

7 Spines

We saw in Section 2 that it is sometimes easier to work with block-tridiagonal matrices rather than with actual growth curves. We want to focus attention now on the block-diagonal matrices that fit inside a given growth curve. They will be among the players needed in section 8 to distinguish the $G(r)$ in terms of bandwidth dimension.

Definition. *Suppose* $0 \le r \le 1$. *A* ***spine*** *for* $G(r)$ *is a subalgebra* S *of* $G(r)$ *for which there is an increasing sequence* $n_1, n_2, \ldots$ *of positive integers such that* S *consists of all matrices of the form*



(In particular $S \cong \prod_{k=1}^{\infty} M_{n_k}(F)$. *Notice that* S *is completely determined by the sequence* $n_1, n_2, \ldots$ *.)* □

Remark. If S is a spine for $G(r)$, then there is a constant $c > 0$ such that cn^r is a growth curve for **all** elements of S. For consider the "fastest growing" element $x \in S$ whose kth diagonal block is

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & & & \\ & & & \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

By definition of $G(r)$, x has $O(n^r)$ growth so there is a constant $c > 0$ for which $f(n) = cn^r$ is a growth curve for x. But now clearly $f(n)$ is also a growth curve for all elements of S.

Clearly $G(r)$ can have many different spines. In §8 we shall need spines that fit the fixed growth curve as closely as possible, and the following result shows us how to construct the block sizes $n_1, n_2, \ldots$ to achieve this goal. Notice that the spine determined by a sequence $\{n_k\}$ of positive integers will be a spine for $G(r)$ if and only if

$$n_{k+1} = O((n_1 + \cdots + n_k)^r).$$

Proposition 7.1. *Suppose $0 \le r \le 1$. Define the sequence $n_1, n_2, \ldots$ as follows :*

(a) *if $r < 1$ then set $t = \dfrac{r}{1-r}$ and let $n_k = [k^t]$* where *$[x]$ denotes the greatest integer less than or equal to x,*

(b) *if $r = 1$ then let $n_k = 2^k$.*

Let S be the subalgebra of $B(F)$ consisting of all matrices of the form shown in the above definition. Then S is a spine for $G(r)$ but is not a spine for $G(s)$ for any $s < r$.

Proof. The Proposition will be proved if we can find positive constants c_1, c_2 such that

$$c_1(n_1 + n_2 + \cdots + n_k)^r \;\le\; n_{k+1} \;\le\; c_2(n_1 + n_2 + \cdots + n_k)^r$$

is true for all large enough k.

The case $r = 1$ is the simpler one (and, indeed, we have already seen the basic idea in the proof of Theorem 2.1). In this case

$$\begin{aligned} n_1 + n_2 + \cdots + n_k &= 2 + 2^2 + \cdots + 2^k \\ &= 2^{k+1} - 2 \\ &= n_{k+1} - 2 \end{aligned}$$

and so $c_1 = 1$ and $c_2 = 2$ will do the trick.

Now suppose $r < 1$ and consider the sequence $n_k = [k^t]$. For the moment we shall just assume that $t > 0$: the reason for the correct value of t will appear during the course of the proof.

Let $f : \mathbf{R}^+ \to \mathbf{R}^+$ be the function $f(x) = x^t$ and notice that f is strictly increasing. Hence for any integer $n \ge 0$ we have

$$[n^t] \;\le\; n^t \;\le\; \int_n^{n+1} f(x)dx \;\le\; (n+1)^t \;\le\; [(n+1)^t] + 1.$$

Adding up all such inequalities for $n = 0, 1, \ldots, k$ gives

$$\sum_{n=1}^{k} [n^t] \;\le\; \int_0^{k+1} f(x)dx \;=\; \frac{(k+1)^{t+1}}{t+1} \;\le\; \Big(\sum_{n=1}^{k+1} [n^t]\Big) + (k+1). \qquad (*)$$

We now use these two inequalities to find suitable values for c_1, c_2. Firstly consider the left-most $\leq$ in $(*)$. We have

$$(k+1)^{t+1} \geq (t+1)\sum_{n=1}^{k} [n^t]$$

and therefore

$$(k+1)^t \geq (t+1)^{t/t+1}\Big\{\sum_{n=1}^{k} [n^t]\Big\}^{t/t+1}$$

Hence for all large enough k we have

$$[(k+1)^t] \geq c_1\Big\{\sum_{n=1}^{k} [n^t]\Big\}^{t/t+1}$$

where c_1 is a suitable positive constant (one such c_1 would be $0.9(t+1)^{t/t+1}$ since $(k+1)^t \to +\infty$ and so $[(k+1)^t] > 0.9(k+1)^t$ eventually). Hence the correct value of t must satisfy $\dfrac{t}{t+1} = r$ or equivalently $t = \dfrac{r}{1-r}$ as claimed in the statement of the Proposition.

Now consider the right-most $\leq$ in $(*)$. From this we get

$$\begin{aligned}
\sum_{n=1}^{k} [n^t] &\geq \frac{(k+1)^{t+1}}{t+1} - [(k+1)^t] - (k+1) \\
&= (k+1)^{t+1}\left\{\frac{1}{t+1} - \frac{[(k+1)^t]}{(k+1)^t}\cdot\frac{1}{k+1} - \frac{1}{(k+1)^t}\right\} \\
&\geq \frac{1}{2(t+1)}\,(k+1)^{t+1}
\end{aligned}$$

for all large enough k, since the second and third terms in $\{\,\cdots\}$ both tend to zero as $k \to \infty$. Hence, using the same value of t as before, we get

$$[(k+1)^t] \leq (k+1)^t \leq c_2\Big\{\sum_{n=1}^{k} [n^t]\Big\}^{r}$$

for all large enough k (where $c_2 = (2(t+1))^r$ is a positive constant). This completes the proof.

8 Algebras with prescribed bandwidth dimension

The principal goal of this section is to establish that for any real number $r \in [0,1]$, and for any field F, there is an algebra A over F of bandwidth dimension r (Theorem 8.8). In fact A can be chosen as a subalgebra of $G(r)$ generated by a suitable spine (isomorphic to some $\prod_{i=1}^{\infty} M_{n_i}(F)$) and two additional elements. A corollary is that $G(r)$ itself has bandwidth dimension r for each $r \in [0,1]$.

In the course of proving Theorem 8.8, we establish some propositions of independent interest. Primarily they concern embeddings of the algebra $R = \prod_{i=1}^{\infty} M_{n_i}(F)$ and $R/\mathrm{soc}R$

in $Q = M_\omega(F)$, for any unbounded increasing sequence $\{n_i\}$ of positive integers. Proposition 8.1 says that all the nonzero singular R–modules have uncountable dimension over F. The significance of this, for us, is that $R/\mathrm{soc}R$ doesn't have a homomorphic image in Q (Proposition 8.2), so that all embeddings of R in Q send complete sets of orthogonal idempotents in R to complete sets ***in Q***. An important consequence of this is that whenever R embeds in some $G(s)$, then the whole of the image of R is bounded by some fixed growth curve $g(n) = cn^s$ for some constant c (Proposition 8.3).

The remaining components in the proof of Theorem 8.8 involve some quite new ideas which relate equivalent orthogonal idempotents and their growth curves, for a subalgebra R of Q. The positioning of the first nonzero rows in these idempotents provides key information. Some of the ideas here may point to other ways in which knowledge of the growth curves for elements of a subalgebra R can be used to derive ring-theoretic properties of R.

Proposition 8.1. *Let F be a field. Let $\{n_i\}_1^\infty$ be any unbounded increasing sequence of positive integers and let*

$$R = \prod_{i=1}^{\infty} M_{n_i}(F).$$

Then all nonzero singular R–modules have uncountable dimension over F.

Proof (courtesy of Ken Goodearl). Notice that the singular R–modules are just the $R/\mathrm{soc}R$ modules. It is clearly enough to prove that all simple singular R–modules have uncountable dimension over F.

Let M be a simple singular R–module. Then the annihilator $A(M)$ of M in R is a primitive and hence prime ideal, and hence by [G3, Corollary 8.23] $A(M)$ contains a minimal prime ideal P of R. By [G1, Proposition 7], there exists an ultrafilter $\mathcal{U}$ on the algebra of subsets of N such that

$$P = \{r \in R \mid \{i \in \mathrm{N} : r_i = 0\} \in \mathcal{U}\}.$$

Since M is a singular R–module, we have $\mathrm{soc}R \subseteq A(M)$. Now P is a prime ideal contained in the proper ideal $A(M)$ and $\mathrm{soc}R$ is generated by central idempotents. Hence $\mathrm{soc}R \subseteq P$. In turn this implies $\mathcal{U}$ contains all cofinite subsets of N. Hence $\mathcal{U}$ is a nontrivial ultrafilter (that is, it is not principal) and so R/P is a nontrivial ultraproduct of the $M_{n_i}(F)$.

Knowing that M is also a module over R/P, we can complete the proof by showing that for any nontrivial ultraproduct S of the $M_{n_i}(F)$, all nonzero S–modules M have uncountable dimension over F. We demonstrate this by showing that S contains a subfield K which is an F–subalgebra of uncountable dimension (and then noting that $\dim_F M = (\dim_K M)(\dim_F K)$). We consider separately the situations where F is infinite or finite.

Suppose F is infinite. Here we choose for K the image of the centre of $\prod M_{n_i}(F)$ in S, which is a nontrivial ultrapower of F over $\mathcal{U}$. Since a nontrivial ultraproduct of infinite sets has cardinality at least $2^{\aleph_0}$ [BS, Proposition, 6.3.14], K is certainly uncountable-dimensional over a countably-infinite F. Suppose F is uncountable. Choose distinct scalar matrices $x_i \in M_{n_i}(F)$ and let $x = (x_1, x_2, \ldots, x_i, \ldots) \in \prod M_{n_i}(F)$. For any nonzero polynomial $f \in F[t]$, the set $\{i \in \mathrm{N} \mid f(x)_i = 0\}$ is finite and hence not in $\mathcal{U}$ because nontrivial ultrafilters contain the cofinite subsets. Hence the image of $f(x)$ in K is nonzero and so invertible, from which we deduce that K contains a copy of the rational function field $F(x)$. For an uncountable F, it is well-known that $F(x)$ has uncountable dimension over F. Therefore so does K.

The remaining case to consider is when F is finite. It is here that we utilize our assumption that the n_i are unbounded and increasing. For each i we choose a finite field extension K_i of F inside $M_{n_i}(F)$ such that the $|K_i|$ are increasing and unbounded. Now for our subfield K of S we take the image of $\prod K_i$ in S, which is a nontrivial ultraproduct of the K_i. Since nontrivial ultraproducts over N of finite sets of unbounded increasing size have cardinality $2^{\aleph_0}$ [BS, Theorem 6.3.12], we again have that K has uncountable dimension over F.

Proposition 8.2. *Let R be a subalgebra of $Q = M_\omega(F)$ and suppose $R \cong \prod_{i=1}^{\infty} M_{n_i}(F)$ for some unbounded increasing sequence $\{n_i\}_1^\infty$ of positive integers. Then Q_R is a nonsingular R–module, that is, $\ell_Q(\mathrm{soc}R) = 0$.*

Proof. Let $J = \mathrm{soc}R$ and let $e \in Q$ be an idempotent such that $\ell_Q(J) = Qe$. (Note that since Q is a regular right self-injective ring, any one-sided anihilator ideal is generated by an idempotent [G2, Proposition 2.9].) For $x \in R$ we have $ex(1-e)J = exJ \subseteq eJ = 0$ implies $ex(1-e) \in \ell_Q(J) = Qe$ and so $ex(1-e) = 0$. Hence $eR(1-e) = 0$. Therefore the map

$$\psi \, : \, R \to eQe \quad , \quad \psi(x) = ex$$

is an algebra homomorphism with $J \subseteq \ker\psi$. Now there exists an algebra isomorphism $eQe \cong \mathrm{End}_F(V)$ for some countable-dimensional vector space ${}_FV$, and therefore we can produce an algebra homomorphism $\theta : R/\mathrm{soc}R \to \mathrm{End}_F(V)$. Thus V becomes an $R/\mathrm{soc}R$ module of countable dimension over F. By Proposition 8.1, $V = 0$. Hence $e = 0$ and $\ell_Q(\mathrm{soc}R) = 0$.

Remark. As the proofs of Propositions 8.1 and 8.2 show, the restriction on the n_i is used only when F is a finite field. In this case, without the restriction, Q_R need not be nonsingular. For example, suppose $F = GF(2)$ and all $n_i = 1$. Let $\mathcal{U}$ be a nontrivial ultrafilter on N. The map

$$\begin{aligned} &\theta \, : \, \textstyle\prod_1^\infty F \to F \\ &\theta(x) = 0 \ \text{ if } \{i \in \mathrm{N} \mid x_i = 0\} \in \mathcal{U} \\ &\theta(x) = 1 \ \text{ otherwise} \end{aligned}$$

is a ring homomorphism (so the corresponding ultraproduct is just F). Hence the set R of all $\omega \times \omega$ diagonal matrices of the form

$$\begin{bmatrix} a_0 & & & & & \\ & a_1 & & & & \\ & & a_2 & & & \\ & & & \ddots & & \\ & & & & a_n & \\ & & & & & \ddots \end{bmatrix}$$

where $a_1, a_2, \ldots, a_n, \ldots$ are arbitrary elements of F and $a_0 = \theta(a_1, a_2, \ldots)$, forms a subalgebra of $Q = M_\omega(F)$ which is isomorphic to $\prod_1^\infty F$. Here soc$R$ consists of those matrices with $a_0 = 0$ and only finitely many $a_i \neq 0$. Hence $\ell_Q(\text{soc}R) = Qe_{11} \neq 0$. Notice that (via θ) F itself is a 1–dimensional simple singular R–module (cf. Proposition 8.1).

Proposition 8.3. *Let R be a subalgebra of $Q = M_\omega(F)$ which is isomorphic to $\prod_{i=1}^\infty M_{n_i}(F)$ for some unbounded increasing sequence $\{n_i\}$ of positive integers. If $R \subseteq G(s)$ for some $s \in [0,1]$, then $R \subseteq W_s(c)$ for some positive c (that is, cn^s is a growth curve for **all** elements of R).*

Proof. Choose a complete set $\{f_n\}_1^\infty$ of central orthogonal idempotents of R such that each f_nR is finite-dimensional over F (take, for example, the idempotent generators for the homogeneous components of soc$R$). Here by ***complete in R*** we mean $\ell_R\{f_n\}_1^\infty = 0$. Note that this is equivalent to soc$R \subseteq \sum f_nR$ because soc$R$ is the smallest essential right ideal, and in a regular right self-injective ring, a right ideal is essential if and only if it has zero left annihilator (see [G2, Propositions 2.9 and 2.11]). By Proposition 8.2, $\ell_Q(\text{soc}R) = 0$ and so $\ell_Q\{f_n\} = 0$. Hence $\{f_n\}_1^\infty$ is also a complete set of orthogonal idempotents ***of Q*** . (This is why Proposition 8.1 is so important to us – getting to this step.)

For the rest of the proof we drop the subscript s in the $W_s(c)$. Let $e_1, e_2, \ldots, e_n, \ldots$ be the standard primitive idempotents of Q (that is, e_i has 1 in the (i,i) position and 0 elsewhere). Notice that for any $x \in Q$, we have $x \in W(c)$ if and only if $xe_i \in W(c)$ for all i. Suppose firstly there is a positive constant c such that $f_nR \subseteq W(c)$ for all but finitely many n. Since the f_nR are finite-dimensional, and $R \subseteq G(s) = \cup_{d\geq 0}W(d)$ with the $W(d)$ forming a chain of F–subspaces (see 1.2), we can adjust c so that $f_nR \subseteq W(c)$ for all n. Now for any $x \in R$ we have

$$\begin{aligned} & xe_i \in \text{soc}Q \subseteq \sum f_nQ && \text{by completeness of } \{f_n\} \text{ in } Q \\ \Rightarrow\ & xe_i = (f_1 + \cdots + f_m)xe_i && \text{for some } m \\ & \quad = (f_1x + \cdots + f_mx)e_i \in W(c) && \text{since } f_jR \subseteq W(c) \\ \Rightarrow\ & xe_i \in W(c) \quad \text{for all } i \\ \Rightarrow\ & x \in W(c). \end{aligned}$$

Hence $R \subseteq W(c)$ in this case.

Now suppose the Proposition is false, so that $R \not\subseteq W(c)$ for any c. Then from the previous paragraph, together with the fact that the f_n are central, we have that for each $k \in \mathbb{N}$ there are infinitely many $Rf_n \not\subseteq W(k)$. This enables us to select an infinite set $\{h_n\}_1^\infty \subseteq \{f_n\}_1^\infty$ of orthogonal idempotents of R such that $Rh_n \not\subseteq W(n)$ for all n. By partitioning $\{h_n\}_1^\infty$ into an infinite number of infinite subsets and taking "sums" over each subset (using injectivity of R_R and the complete set $\{f_n\}_1^\infty$) we can produce an infinite set $\{g_n\}_1^\infty$ of orthogonal central idempotents of R such that each Rg_n contains infinitely many h_m. (For $H \subseteq \{f_n\}_1^\infty$, the "sum" of the elements in H is the unique (idempotent) element $h \in R$ satisfying $hf_n = f_n$ for all $f_n \in H$, $hf_n = 0$ for all $f_n \notin H$.) Now the important property of the g_n is that

$$Rg_n \not\subseteq W(d) \qquad \text{for any positive real number } d.$$

For if $Rg_n \subseteq W(d)$ we could choose a positive integer $m \geq d$ such that $h_m \in Rg_n$, which would yield the contradiction

$$Rh_m \subseteq Rg_n \subseteq W(d) \subseteq W(m).$$

The remainder of the proof parallels that for the second half of 5.1. Firstly we choose a strictly increasing sequence $\{k_n\}$ of positive integers such that $g_n \in W(k_n)$, which is possible because $R \subseteq G(s)$. Next, since from above $Rg_n \not\subseteq W(k_n^2)$, we can choose $x_n \in Rg_n$ with $x_n \notin W(k_n^2)$. By injectivity of R_R, there is an $x \in R$ satisfying $xg_n = x_n$ for all n. Moreover $x \in W(m)$ for some positive m, again because $R \subseteq G(s)$. By 1.2(b), $x_n = xg_n$ implies $x_n \in W(m)W(k_n) \subseteq W(m + k_n + mk_n)$ because $s \leq 1$. For large n, $W(m + k_n + mk_n) \subseteq W(k_n^2)$ which implies $x_n \in W(k_n^2)$. This is a contradiction. We conclude that $R \subseteq W(c)$ for some c.

The next example shows that Proposition 8.3 does not hold for a general regular self-injective subalgebra R of $M_\omega(F)$, even for R a field. Thus, although our proof of 8.3 (via 8.1 and 8.2) may appear rather circuitous, this example suggests that a substantially shorter proof may not be possible (that is, a proof which avoids the question of whether $R/\mathrm{soc}R$ has a homomorphic image in Q).

Example 8.4. Let $F = GF(2)$. By considering the regular representation of $GF(2^{n+1})$ in terms of 2×2 matrices over $GF(2^n)$, we can construct for each $n \in \mathbb{N}$ a copy K_n of $GF(2^n)$ inside $M_{2^n}(F)$ in such a way that

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} \in K_{n+1} \qquad \text{for all } B \in K_n,$$

and such that some member of K_{n+1} has an invertible off-diagonal entry when viewed as a 2×2 matrix over $M_{2^n}(F)$. Then some member of K_{n+1} must have bandwidth at least 2^n when viewed as a $2^{n+1} \times 2^{n+1}$ matrix over F. Now consider the algebra R of all $\omega \times \omega$ matrices over F of the form

$$\begin{bmatrix} B & & & O \\ & B & & \\ & & B & \\ & & & \ddots \\ & O & & \end{bmatrix}$$

where B is a $2^n \times 2^n$ matrix in K_n for some n. Then $R \cong \varinjlim GF(2^n)$ and $R \subseteq G(0)$. But because there is no fixed finite bandwidth for all members of R, we see that $R \not\subseteq W_0(c)$ for any positive c. Hence the conclusion of Proposition 8.3 is not valid here. $\square$

There is nothing special about having F finite in this example (unlike the earlier example). In fact we could take any field F which has a countably-infinite dimensional algebraic extension R. It is not hard to construct slightly more complicated examples of algebras R where 8.3 fails but where R is an infinite direct product of simple Artinian algebras. Here 8.3 fails not because of the lack of orthogonal idempotents, but because some of the simple Artinian factors are not finite-dimensional.

For a general ring R and elements $a, b \in R$, we coin the term a ***cross-element from bR to aR*** to mean any $\gamma \in aRb$ such that $\gamma R = aR$ and $R\gamma = Rb$. Notice that if a and b are idempotents with $bR \cong aR$, then any $\gamma \in aRb$ which induces this isomorphism under left multiplication is a cross-element. Notice too that a cross-element from bR to aR is automatically a cross-element from bS to aS for any overring S of R.

Lemma 8.5. *Let $Q = M_\omega(F)$ and let $a, b \in B(F)$ be nonzero. Suppose γ is a cross-element from bQ to aQ such that a, b, and γ all have a common growth curve $f(n)$ which is increasing. Let ℓ (respectively m) be the row index of the first nonzero row of a (respectively b). Then*

$$m - \ell \leq f(\ell) + f(\ell + [f(\ell)])$$

if $m \geq \ell$, with a similar result when $m \leq \ell$.

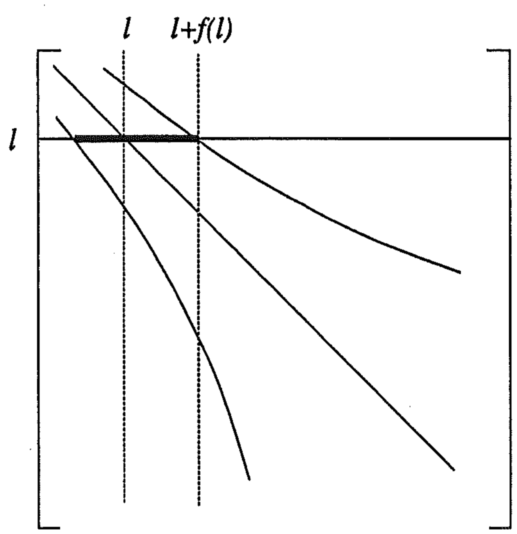
Proof. Suppose $m \geq \ell$. Since γ is a cross-element from bQ to aQ we have $\gamma Q = aQ$, and so γ and a have nonzero rows in the same positions. Hence ℓ is also the row index of the first nonzero row of γ. We now estimate how big m can be in terms of ℓ.

Notice firstly that we also have $Q\gamma = Qb$ and so γ and b have nonzero columns in the same positions. Since $f(n)$ is a growth curve for γ the nonzero entries in the ℓth row of γ must occur among the jth columns where $j \leq \ell + [f(\ell)]$. See the figure below. Hence b must have a nonzero jth column for some $j \leq \ell + [f(\ell)]$. But $f(n)$ is also a growth curve for b, and so for any j the nonzero entries in the jth column of b must occur in an ith row where $i \le j + f(j)$. Since the mth row is the first nonzero row of b we thus have

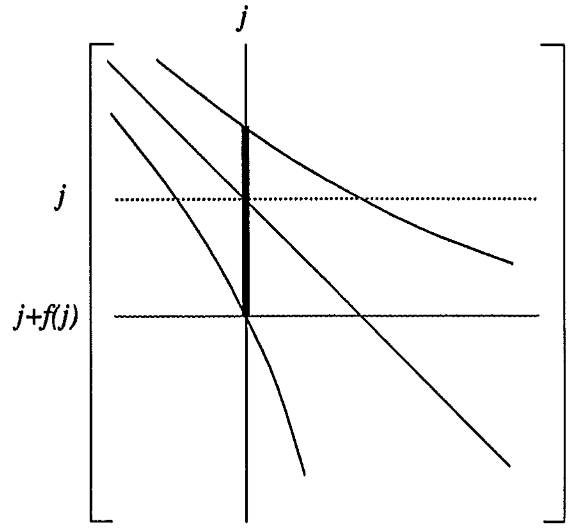
$$\begin{aligned} m &\le \max\{j + f(j) : j \le \ell + f(\ell)\} \\ &\le \ell + f(\ell) + f(\ell + [f(\ell)]) \end{aligned}$$

as required.

A similar proof works for the case $m \le \ell$, where we would need the fact that $f(n)$ is a growth curve for a.



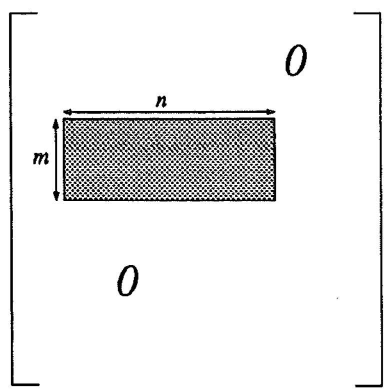
γ



b

Remark. Notice that the proof of Lemma 8.5 uses both the upper and lower bounds provided by the growth curve (the upper curve is being used in the diagram for γ, while the lower curve is used in the diagram for b).

Lemma 8.6. *Let m and n be positive integers and let Z be some fixed $m \times n$ rectangular block within $Q = M_\omega(F)$, that is, Z consists of all matrices whose nonzero entries are within the shaded area:*



(Notice that we do not require any special placement of the rectangle.) Let $\{g_i\}_{i\in I}$ *be any set of orthogonal idempotents of* Q*. Then at most* n *of the* g_i *can have* **some** *nonzero row entirely within* Z*.*

Proof. Note that Z is a (unitary) left module over $B = M_m(F)$ of uniform (Goldie) dimension n. Suppose (after relabelling) that $g_1, g_2, \ldots, g_{n+1}$ each have a nonzero row entirely within Z, say

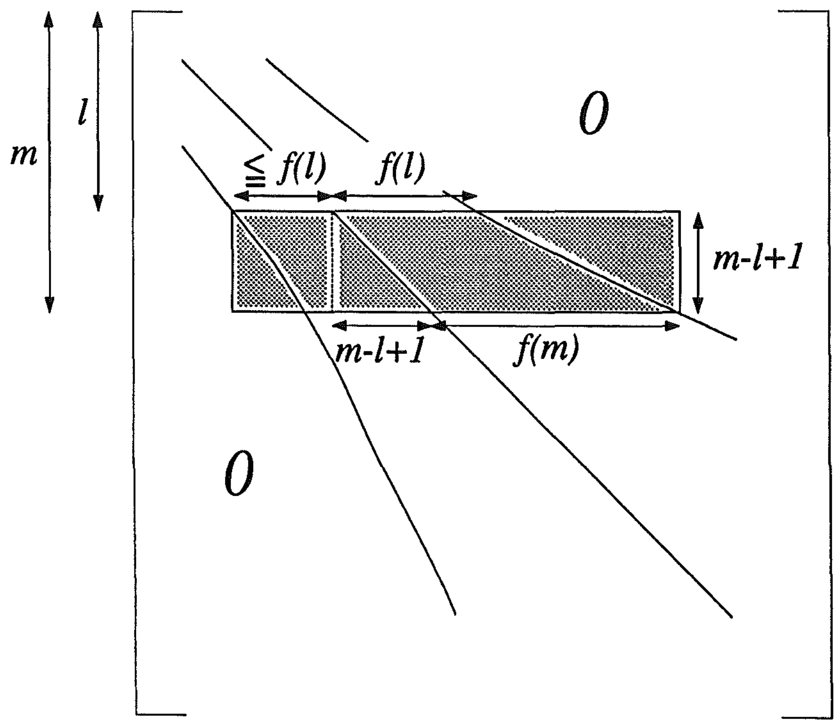
$$0 \neq h_i g_i \in Z$$

for a suitable (standard) primitive idempotent $h_i \in Q$. Then the Bh_ig_i are nonzero independent left B–submodules of Z for $i = 1, \ldots, n+1$, which is impossible because ${}_BZ$ has uniform dimension n.

Lemma 8.7. *Let* $g_1, \ldots, g_k \in B(F)$ *be orthogonal idempotents and suppose* $f(n)$ *is an increasing growth curve for these idempotents. Suppose* ℓ *and* m *are positive integers with* $\ell \leq m$ *and such that each* g_i *has some nonzero row between the* ℓ^{th} *and* m^{th} *(inclusive). Then*

$$k \;\leq\; 2f(m) + m - \ell + 1.$$

Proof. By the very nature of a growth curve, all the g_i have some nonzero row entirely within the shaded area of the figure below.



Since $f(n)$ is increasing, this rectangular block has

$$\mathit{width} \ \leq\ f(\ell) + (m - \ell + 1) + f(m) \ \leq\ 2f(m) + m - \ell + 1.$$

Hence by Lemma 8.6, we have $k \ \leq\ 2f(m) + m - \ell + 1$.

Remark. The above argument again utilizes the fact that $f(n)$ is both an upper and a lower growth curve.

Theorem 8.8. *Let r be any real number between 0 and 1, and let F be any field. Then there exists an algebra A over F of bandwidth dimension r. In fact A can be chosen as a subalgebra of $G(r)$ generated by a suitable spine and two other elements.*

Proof. Let $r \in [0, 1]$. The case $r = 0$ is trivial and the case $r = 1$ is taken care of by 3.2, 2.3, and 2.1. (For the first part of the Theorem when $r = 1$, we only need the fact that countable-dimensional, purely infinite algebras have bandwidth dimension 1 (see 3.3).) Henceforth we assume that $0 < r < 1$. The following are fixed throughout the proof:

Notation. Let

$R \cong \prod_{k=1}^{\infty} M_{n_k}(F)$ be the spine of $G(r)$ determined by $n_k = [k^t]$ for $t = \frac{r}{1-r}$ (see Proposition 7.1).

J_k = k^{th} homogeneous component of $\mathrm{soc}R$, that is, J_k consists of all matrices in R whose entries are 0 except in the k^{th} diagonal block (so that $J_k \cong M_{n_k}(F)$).

F_k = a chosen set of n_k orthogonal primitive idempotents of J_k (e.g. the standard ones).

Q = $M_\omega(F)$.

$$x = \begin{bmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{bmatrix}, \quad y = \begin{bmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & \end{bmatrix}$$

are the standard shift matrices in Q.

A = the subalgebra of $B(F)$ generated by R, x, y.

$\theta : A \to G(s)$ is an algebra embedding for some $0 \leq s \leq r$.

$W_s(c) = \{x \in B(F) \mid cn^s \text{ is a growth curve for } x\}$ for any constant c.

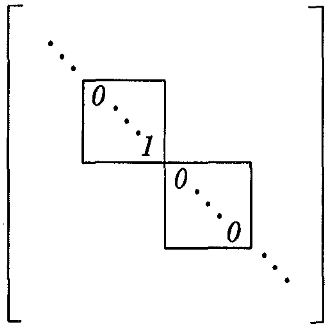
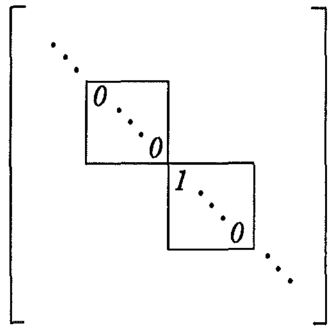
Clearly the theorem is proven once we show that $s \geq r$. This we will achieve in a series of steps. Notice that by Theorem 5.1 we know $s > 0$ because A contains the spine R which is a regular self-injective ring of unbounded index of nilpotence.

Our strategy is to chase the first nonzero rows of the images of the idempotents in F_k, and obtain opposing constraints on their positions. On the one hand they have to be fairly close together to ensure that cross-elements from $\theta(F_k)$ to $\theta(F_{k+1})$ lie in $G(s)$. On the other hand, having the n_k images of the equivalent orthogonal idempotents from F_k all inside $W_s(c)$ forces their nonzero rows to become increasingly scattered.

Step 1. *There is a constant $c > 0$ such that $W_s(c)$ contains $\theta(R)$ and such that, for each $k \geq 1$ and for any pair of primitive idempotents $e \in J_k$ and $f \in J_{k+1}$, there are cross-elements in $W_s(c)$ from $\theta(f)Q$ to $\theta(e)Q$ and also from $\theta(e)Q$ to $\theta(f)Q$.*

Proof. By Proposition 8.3 there is some constant $c_1 > 0$ such that $\theta(R) \subseteq W_s(c_1)$, and by increasing c_1 if necessary, we can also assume that $\theta(x), \theta(y) \in W_s(c_1)$.

Fix k, and consider any pair of primitive idempotents $e \in J_k$ and $f \in J_{k+1}$. Let g be the last standard primitive idempotent of J_k, and let h be the first one in J_{k+1}. (See figure below.)

$g =$  $h =$ 

Since e, g are both primitive idempotents in J_k we have $eJ_k \cong gJ_k$ and so there are elements $\alpha_1 \in gJ_k e$ and $\gamma_1 \in eJ_k g$ such that $\alpha_1\gamma_1 = g$ and $\gamma_1\alpha_1 = e$ (by [Jac 2, Proposition 4, page 51]). Similarly there are elements $\alpha_2 \in fJ_{k+1}h$ and $\gamma_2 \in hJ_{k+1}f$ such that $\alpha_2\gamma_2 = f$ and $\gamma_2\alpha_2 = h$. Notice also that $xg \in hAg$ and $yh \in gAh$, and that these elements satisfy $(xg)(yh) = h$ and $(yh)(xg) = g$. To get the desired cross-elements we essentially glue together the obvious isomorphisms

$$eA \to gA \to hA \to fA$$

and their inverses. Thus it is easy to check that $\alpha_2(xg)\alpha_1 = \alpha_2 x \alpha_1$ is a cross-element from eA to fA, and so that $\beta = \theta(\alpha_2 x \alpha_1)$ is a cross-element from $\theta(e)Q$ to $\theta(f)Q$. Similarly $\gamma = \theta(\gamma_1 y \gamma_2)$ is a cross-element from $\theta(f)Q$ to $\theta(e)Q$. Now $\beta = \theta(\alpha_2)\theta(x)\theta(\alpha_1) \in W_s(c_1)^3$ and similarly $\gamma \in W_s(c_1)^3$. But by Proposition 1.2 there is some constant $c > 0$ (independent of k, e or f) such that $W_s(c_1)^3 \subseteq W_s(c)$. Since the cross-elements $\beta, \gamma \in W_s(c)$ we thus have the desired result.

For each $k \in \mathbf{N}$, we choose an idempotent $f_k \in F_k$ such that $\theta(f_k)$ has its first nonzero row at least as far down as for any of the other $n_k - 1$ idempotents in F_k. We then set

$$p_k = \text{row index of the } \textbf{\textit{first}} \text{ nonzero row of } \theta(f_k).$$

Step 2. $\quad p_k \geq \dfrac{1}{(1+c)}\left(\dfrac{n_k}{5c}\right)^{1/s}$ *for all* k.

Proof. Fix a positive integer k. Let ℓ be the smallest row index of any nonzero row of any of the images $\theta(g)$, as g ranges over the idempotents in F_k. Suppose this smallest index occurs for $\theta(g_k)$. Let $m = p_k$ $(\geq \ell)$. By Step 1, $\theta(R) \subseteq W_s(c)$ so we have a cross-element γ from $\theta(f_k)Q$ to $\theta(g_k)Q$ inside the increasing growth curve $f(n) = cn^s$. Hence by Lemma 8.5

$$\begin{aligned} m - \ell &\leq f(\ell) + f(\ell + [f(\ell)]) \\ &\leq f(m) + f(m + [f(m)]) \\ &\leq 2f(m + [f(m)]). \end{aligned}$$

By definitions of ℓ and m, all the n_k images $\theta(g)$ of the idempotents $g \in F_k$ have a nonzero row between the ℓ^{th} and m^{th}. Therefore by Lemma 8.7

$$\begin{aligned} n_k &\leq 2f(m) + m - \ell + 1 \\ &\leq 2f(m) + 2f(m + [f(m)]) + 1 && \text{(from above)} \\ &\leq 5f(m + [f(m)]) \\ &\leq 5c(p_k + cp_k^s)^s \\ &\leq 5c(1+c)^s p_k^s && \text{(since } s \leq 1) \end{aligned}$$

which implies (since $0 < s$) that

$$p_k \geq \frac{1}{(1+c)}\left(\frac{n_k}{5c}\right)^{1/s}$$

Step 3. $\quad |p_{k+1} - p_k| \leq 2c(1+c)^s p_k^s$ *for all* k.

Proof. Firstly consider the case $p_{k+1} \geq p_k$. By Step 1 there is a cross-element γ from $\theta(f_{k+1})Q$ to $\theta(f_k)Q$ with a growth curve $f(n) = cn^s$, whence by Lemma 8.5

$$\begin{aligned} p_{k+1} - p_k &\leq f(p_k) + f(p_k + [f(p_k)]) \\ &\leq 2f(p_k + [f(p_k)]) \\ &\leq 2c(p_k + cp_k^s)^s \\ &\leq 2c(1+c)^s p_k^s && \text{since } s \leq 1\,. \end{aligned}$$

The other case, $p_k \geq p_{k+1}$, is similar. For by Step 1 there is a cross-element γ from $\theta(f_k)Q$ to $\theta(f_{k+1})Q$ inside $f(n)$, so again by Lemma 8.5 we have

$$\begin{aligned} p_k - p_{k+1} &\leq f(p_{k+1}) + f(p_{k+1} + [f(p_{k+1})]) \\ &\leq f(p_k) + f(p_k + [f(p_k)]). \end{aligned}$$

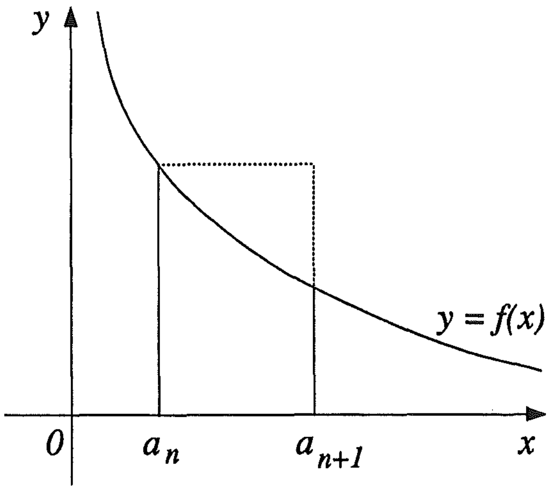
The proof now proceeds exactly as before.

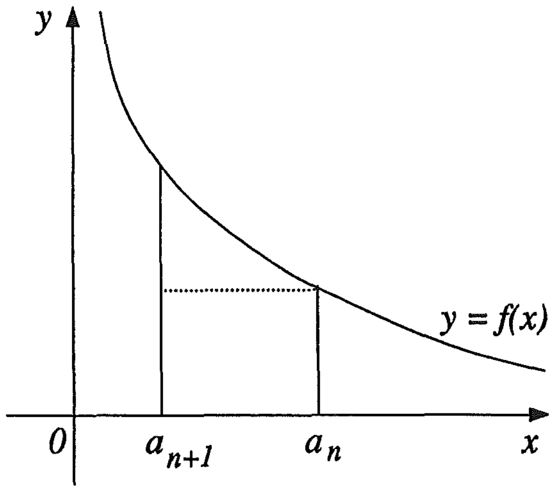
Step 4. *Suppose* $0 < s < 1$ *and* $\{a_k\}$ *is a sequence of positive numbers. If there is a positive constant* c_1 *such that* $|a_{k+1} - a_k| \leq c_1 a_k^s$ *for all* $k \geq 1$, *then there is a positive constant* c_2 *such that* $a_k \leq c_2 k^{1/(1-s)}$.

Proof. Let $f : \mathbf{R}^+ \to \mathbf{R}^+$ be the function $f(x) = x^{-s}$ and notice that f is a decreasing function since $s > 0$. We claim that for each $n \geq 1$

$$\frac{a_{n+1} - a_n}{a_n^s} \geq \int_{a_n}^{a_{n+1}} f(x)dx \qquad (*)$$

Indeed if $a_{n+1} > a_n$ then the situation is illustrated in the left-hand diagram below.





Hence in this case

$$\frac{a_{n+1} - a_n}{a_n^s} = \text{area of rectangle} \geq \int_{a_n}^{a_{n+1}} f(x)dx.$$

On the other hand if $a_{n+1} < a_n$ then the right-hand diagram illustrates the situation, and we get

$$\begin{aligned} \frac{a_{n+1} - a_n}{a_n^s} &= (-1) \times \text{area of rectangle} \\ &\geq -\int_{a_{n+1}}^{a_n} f(x)dx = \int_{a_n}^{a_{n+1}} f(x)dx. \end{aligned}$$

Now adding the inequalities $(*)$ for $n = 1, 2, \ldots, k-1$ and using the given inequality, we get

$$
\begin{aligned}
(k-1)c_1 = \sum_{n=1}^{k-1} c_1 &\geq \sum_{n=1}^{k-1} \frac{|a_{n+1} - a_n|}{a_n^s} \geq \sum_{n=1}^{k-1} \frac{a_{n+1} - a_n}{a_n^s} \\
&\geq \sum_{n=1}^{k-1} \int_{a_n}^{a_{n+1}} f(x)dx = \int_{a_1}^{a_k} f(x)dx \\
&= \frac{a_k^{1-s} - a_1^{1-s}}{1-s}.
\end{aligned}
$$

Hence $a_k^{1-s} \leq c_1(1-s)k + \{a_1^{1-s} - c_1(1-s)\}$ and taking $(1-s)$th roots gives the desired result.

Remark. It is not difficult to see that we cannot reduce the index $\frac{1}{1-s}$ in the conclusion of Step 4. Indeed if we consider the sequence $n_k = [k^t]$, where $t = \frac{s}{1-s}$, and let $a_k = n_1 + n_2 + \cdots + n_k$, then the calculations in the proof of Proposition 7.1 show that the sequence $\{a_k\}$ satisfies the hypothesis of Step 4 and that $a_k \geq ck^{1/(1-s)}$ for some constant $c > 0$.

We can now complete the proof of Theorem 8.8 by comparing the contrasting estimates for the growth of the p_k which we found in Steps 2, 3 and 4. Remarkably, despite all the approximations in our earlier calculations, the numbers fall out exactly the way we want them!

By Step 2 we know that for all k

$$p_k \geq \frac{1}{1+c}\left(\frac{n_k}{5c}\right)^{1/s}.$$

Recalling that $n_k = [k^t]$ where $t = \frac{r}{1-r}$, we see that there is a constant $c_1 > 0$ such that eventually

$$p_k \geq c_1 k^{t/s}.$$

On the other hand, by Step 3 and Step 4 there is another constant $c_2 > 0$ such that for all k

$$p_k \leq c_2 k^{1/(1-s)}.$$

Comparing these opposing growths we conclude that

$$\frac{1}{1-s} \geq \frac{t}{s} = \frac{r}{(1-r)s}$$

and so since $r, s \in (0,1)$ we must have $s \geq r$.

Corollary 8.9. *For each* $r \in [0,1]$, *the algebra* $G(r)$ *has bandwidth dimension* r.

Proof. By definition $G(r)$ has bandwidth dimension at most r, and by Theorem 8.8, $G(r)$ cannot be embedded in $G(s)$ for any $s < r$. Hence $G(r)$ has bandwidth dimension r.

We return to a question raised in the Introduction: what are the possible bandwidth dimensions for countable-dimensional algebras? In the light of the steps used in the proof of Theorem 8.8, we make the following conjecture.

Conjecture 8.10. *For any field F and for any real number r in $[0,1]$, there exists a finitely generated algebra A over F of bandwidth dimension r.*

If the conjecture were true, then by Theorem 2.1 the bandwidth dimensions of countable-dimensional algebras would exactly fill $[0,1]$.

One possible starting point for the construction of A is to take the standard matrix representation of the 4–generator algebra in 6.3. Many of the steps used in the proof of 8.8 would carry over to this algebra. The sticking point is getting the appropriate analogue of Step 1. Although the standard shifts x and y generate the "finite-dimensional fragments" of the spine R, that is $\mathrm{soc}R$, there seems no way of ensuring that $\theta(\mathrm{soc}R)$ is inside some fixed growth curve $f(n) = cn^s$. Thus we would need to allow for a growth of the "constant" c in Steps 2 and 3.

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Stop Press (May 1992)

Since this paper went to Press, we can report that we have now shown that Conjecture 8.10 is indeed true! Thus the bandwidth dimensions of finitely generated algebras exactly fill the interval $[0,1]$. (Prior to this, all our examples of algebras of bandwidth dimension $r \in (0,1)$ had been uncountable-dimensional.)

Our present proof of 8.10 (for $0 < r < 1$) is quite long, some 20 pages, but hopefully it can be simplified. The generators for the algebra $A \subseteq G(r)$ (there are 8 of them) are much more intricate than those in 6.3. Following on from the suggestion in the final paragraph on p.40, the constant c in Steps 2 and 3 is replaced by a general c_k, one for each $k \geq 1$: thus for the given embedding $\theta : A \to G(s)$, $c_k n^s$ is a suitably chosen growth curve for elements in $\theta(F_k)$, for suitable cross-elements between each pair, and also for suitable cross-elements between elements of $\theta(F_k)$ and $\theta(F_{k+1})$. Steps 2, 3 and 4 then lead to the relationship

$$n_k = O\left(k^{s/1-s} c_k^{(1+s)/(1-s)}\right)$$

One of the key ideas in the proof of the Conjecture is the construction of generators for A in such a way that the number of products required to produce each of the standard matrix units in the kth block $J_k \cong M_{n_k}(F)$ grows essentially logarithmically in k — in fact we get by with $O((\log k)^2)$ growth. Then (because $s < 1$) the c_k can be chosen such that

$$c_k = O\left((\log k)^{2/(1-s)}\right)$$

But now since n_k has polynomial growth k^t (for $t = r/(1-r)$), we must have $t \leq s/(1-s)$, and so $r \leq s$ as before.

Another key ingredient in the proof is the observation that powers of the subspaces $W_s(c)$, defined in 1.2, actually grow "at most polynomially" when $s < 1$:

$$(W_s(c))^m \subseteq W_s(dm^{1/1-s})$$

for some positive constant d. This throws up an important distinction between sublinear growth and linear growth, because for $s = 1$ the powers of $W_s(c)$ have genuine exponential growth.